Chapter 2: Matrix

1. Definition
2. Trace, and inversion
3. Transpose
4. Diagonalization and eigenvalue problems
5. Hermitian, unitary, and normal matrices

Definition

An array of numbers (functions)

$$\mathbf{A}=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Ex. rotational matrix

$$\mathbf{U}=\begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Linear equations:

$$\begin{aligned} a_{11}x_1+a_{12}x_2+a_{13}x_3&=b_1 \\ a_{21}x_1+a_{22}x_2+a_{23}x_3&=b_2 \\ a_{31}x_1+a_{32}x_2+a_{33}x_3&=b_3 \end{aligned}$$

$$\begin{aligned} \mathbf{a}&=(a_{11} \quad a_{12} \quad a_{13}) \\ \mathbf{b}&=(a_{21} \quad a_{22} \quad a_{23}) \\ \mathbf{c}&=(a_{31} \quad a_{32} \quad a_{33}) \end{aligned}$$

$$\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}=0?$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

- $\det\mathbf{A}=0$

  homogeneous: nontrivial
  Inhomogeneous: no solution

- $\det\mathbf{A}\neq 0$

  homogeneous: trivial
  Inhomogeneous: unique solution

Properties

1. Equality
   A=**B** => $a_{ij}=b_{ij}$
2. Addition
   1. Associativity (**A**+**B**)+**C**=**A**+(**B**+**C**)
   2. Commutativity **A+B=B+A**
   3. Distributivity a*(**A+B**)=a*A+a*B; (a+b)*A=a*A+a*B
   4. Null matrix (identity element of addition)
      A+**0**=**A** for any **A**
3. Scalar multiplication
   1. Compatibility a*(b*A)=(a*b)*A
   2. Identity element 1*A=A
4. Multiplication
   1. Not commutative in general
   2. Direct product $\mathbf{A} \otimes \mathbf{B} = \mathbf{C}$
5. Inverse matrix
   $\mathbf{A}\mathbf{A}^{-1}=\mathbf{I}$

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

$$|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$$

Trace

Diagonal Matrix: all nondiagonal elements are zero.

$$\mathbf{A} = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$$

Trace: the sum of the diagonal elements

$$Trace(\mathbf{A}-\mathbf{B}) = Trace(\mathbf{A}) - Trace(\mathbf{B})$$

$$Trace(\mathbf{AB}) = \sum_I (\mathbf{AB})_{ii} = \sum_i \sum_j a_{ij} b_{ji} =$$

$$\sum_j \sum_i b_{ji} a_{ij} = \sum_j (\mathbf{BA})_{jj} = Trace(\mathbf{BA})$$

$$Trace([\mathbf{A},\mathbf{B}]) = Trace(\mathbf{AB}-\mathbf{BA}) = 0$$

Inversion

Matrix Inversion $$AA^{-1} = A^{-1}A = 1$$

With $$(A^{-1})_{ij} \equiv a_{ij}^{(-1)}$$ $$a_{ij}^{(-1)} \equiv \frac{C_{ij}}{|A|}$$

Assumption that $A, |A| \neq 0.$

If $M_L A = 1$ then $M_L = A^{-1}$. For example :

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{pmatrix} \qquad A = \begin{pmatrix} 3 & 2 & 1 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & ? & ? & ? \\ 0 & 1 & 0 & ? & ? & ? \\ 0 & 0 & 1 & ? & ? & ? \end{pmatrix} \qquad A^{-1} = \begin{pmatrix} \frac{11}{18} & \frac{-7}{18} & \frac{-1}{18} \\ \frac{-7}{18} & \frac{11}{18} & \frac{-1}{18} \\ \frac{-1}{18} & \frac{-1}{18} & \frac{5}{18} \end{pmatrix}$$

Quiz

1. **A** and **B** are anticommute. $\mathbf{A}^2 = \mathbf{B}^2 = 1$. Show trace(**A**)=trace(**B**)=0.

2. The matrix equation $A^2=0$ does not imply A=0. Show that the most general 2x2 matrix whose square is zero can be written by

$$\begin{pmatrix} ab & b^2 \\ -a^2 & -ab \end{pmatrix}$$

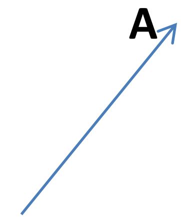
3. If C=A+B, in gerneral $\det C \neq \det A + \det B$

4. Find the inverse of $\begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 4 \end{pmatrix}$

Rotation

Take the 2D space for simplicity, for any vector **A**, its new representation in a rotated axes is given by

$$\begin{bmatrix} A'_x \\ A'_y \end{bmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{bmatrix} A_x \\ A_y \end{bmatrix}$$



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Rotate the vector instead of the coordinate

$$\begin{bmatrix} A_x \\ A_y \end{bmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^{-1} \begin{bmatrix} A'_x \\ A'_y \end{bmatrix}$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{bmatrix} A'_x \\ A'_y \end{bmatrix}$$

Orthogonal matrix

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$UU^{-1} = I$$

Orthogonal condition: $\sum_i a_{ij} a_{ik} = \delta_{ik}$

Transpose

Transpose Matrix

Definition:

$$\widetilde{A} : \widetilde{a}_{ji} = a_{ij}$$

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad \widetilde{\mathbf{A}} = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

For orthogonal matrix:

$$\widetilde{A}A = 1$$

$$\sum_i a_{ij} a_{ik} = \delta_{jk}$$

$$\sum_i a_{ji} a_{ki} = \delta_{jk}$$

$$\Rightarrow \quad \left. \begin{aligned} \widetilde{A}A = A\widetilde{A} = 1 \\ A^{-1}A = 1 \end{aligned} \right\} \Rightarrow \widetilde{A} = A^{-1}$$

Diagonalization

Momentum of inertia matrix:

$$\mathbf{L} = \mathbf{I}\boldsymbol{\omega}$$

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

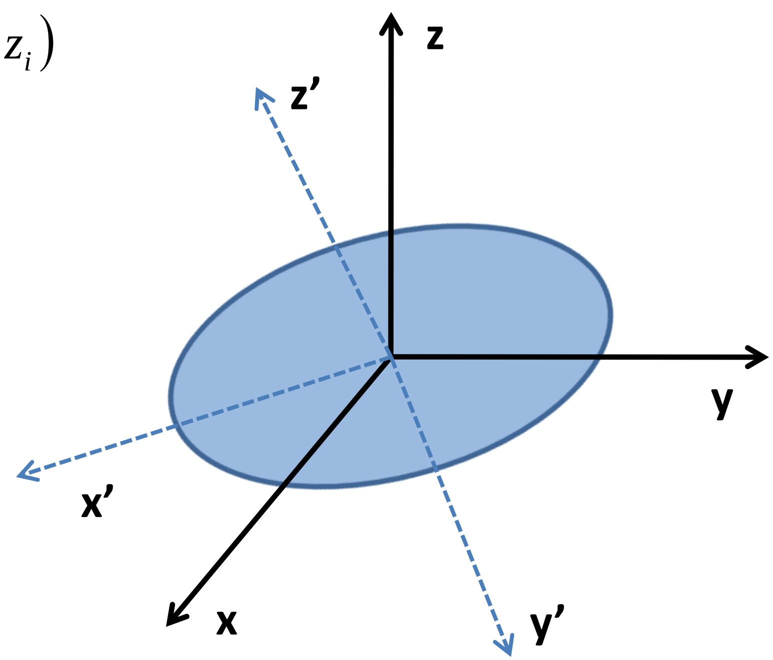
$$I_{xx} = \sum_i m_i \left(r_i^2 - x_i^2\right)$$

$$I_{xy} = -\sum_i m_i x_i y_i$$

$$\mathbf{r}_i = (x_i, y_i, z_i)$$

In the body fixed frame:

$$\begin{pmatrix} L_{x'} \\ L_{y'} \\ L_{z'} \end{pmatrix} = \begin{pmatrix} I_{x'x'} & 0 & 0 \\ 0 & I_{y'y'} & 0 \\ 0 & 0 & I_{z'z'} \end{pmatrix} \begin{pmatrix} \omega_{x'} \\ \omega_{y'} \\ \omega_{z'} \end{pmatrix}$$



$$\langle n|\mathbf{I}|n\rangle = I$$

$$|n'\rangle = \mathbf{U}|n\rangle \Rightarrow \langle n'|\mathbf{I}'|n'\rangle = \langle n|\tilde{\mathbf{U}}\mathbf{I}'\mathbf{U}|n\rangle$$

$$\tilde{\mathbf{U}}\mathbf{I}'\mathbf{U} = \mathbf{I} = \mathbf{U}^{-1}\mathbf{I}'\mathbf{U} \quad \text{or} \quad \mathbf{I}' = \mathbf{U}\mathbf{I}\mathbf{U}^{-1} \qquad \mathbf{U} = ?$$

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Eigenvalues

Momentum of inertia matrix:

$$\tilde{\mathbf{U}}\mathbf{I}' = \mathbf{I}\tilde{\mathbf{U}} \qquad \tilde{\mathbf{U}} = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$$

$$\mathbf{I}\mathbf{v}_\mathrm{i} = \mathbf{I}'_\mathrm{i}\mathbf{v}_\mathrm{i}$$

$$(\mathbf{I} - \lambda\mathbf{1})\mathbf{v} = 0$$

Secular equation:

$$|\mathbf{I} - \lambda\mathbf{1}| = 0$$

$$\begin{vmatrix} I_{xx} - \lambda & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} - \lambda & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} - \lambda \end{vmatrix} = 0$$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \tilde{U} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda = ?$$

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Hermitian, Unitary

Hermitian Matrix, Unitary Matrix

1. Complex conjugate, $\mathbf{A}^*$, formed by taking the complex conjugate of each element, where $i = \sqrt{-1}$

2. Adjoint , **A†** , formed by transposing **A***, **A†**= $\tilde{\mathbf{A}}^*$

3. Hermitian matrix: The matrix A is labeled Hermitian (or self-adjoint)if **A = A†**
   If A is real, then **A†** = $\tilde{\mathbf{A}}$ and real Hermitian matrices are real symmetric matrices.

4. Unitary matrix: Matrix U is labeled unitary if
   U† = U^{-1}

   If **U** is real, then U† = $\tilde{U}$ so real unitary matrices are orthogonal matrices.

5. $(AB)^* = A^* B^*$, **(AB) † = B†A†**

Hermitian

Eigenvalue equations:

$$\mathbf{A}|\mathbf{r}\rangle = \lambda|\mathbf{r}\rangle$$

Real eigenvalues and orthogonal eigenvectors

$$\mathbf{A}|\mathbf{r}_i\rangle = \lambda_i|\mathbf{r}_i\rangle$$
$$\mathbf{A}|\mathbf{r}_j\rangle = \lambda_j|\mathbf{r}_j\rangle$$

$$(\lambda_i - \lambda_j^*)\langle\mathbf{r}_j|\mathbf{r}_i\rangle = 0$$

Ex. Degenerate eigenvalues

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \tilde{U} = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

$$\lambda = -1,1,1$$

Anti-Hermitian matrix: **A =- A†**

If Hermitian matrix A to be diagonalized by unitary matrix U, we have:

$$\det(\exp(\mathbf{A})) = \exp(tr(\mathbf{A}))$$

Normal

Normal matrix: **[A, A†]= AA†-A†A=0**

Hermitian, Anti-Hermitian, Unitary matrices are all normal matrices.

Orthogonal eigenvectors

1. **A** and **A†** have the same eigenvectors with the eigenvalues are complex conjugated to each other.

$$\mathbf{A}|\mathbf{r}\rangle = \lambda|\mathbf{r}\rangle$$

$$\mathbf{A}^{\oplus}|\mathbf{r}\rangle = \lambda^*|\mathbf{r}\rangle$$

2. For any different eigenvalues, the corresponding eigenvectors are orthogonal.

$$\mathbf{A}|\mathbf{r}_i\rangle = \lambda_i|\mathbf{r}_i\rangle$$

$$\mathbf{A}|\mathbf{r}_j\rangle = \lambda_j|\mathbf{r}_j\rangle$$

$$\langle\mathbf{r}_i|\mathbf{A}|\mathbf{r}_j\rangle = ?$$

$$(\lambda_i - \lambda_j)\langle\mathbf{r}_j|\mathbf{r}_i\rangle = 0$$