Chapter 3: Group Theory

1. Definition
2. Mapping
3. Representation
4. Continuous group
5. Lie group and Lie algebra
6. Discrete group

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Definition

A set of objects or operations, rotations, transformations, called the elements of G, that may be combined, or "multiplied", to form a well-defined product in G, that satisfies the following four conditions:

1. Closure

$$\forall a,b \in G, a*b \in G$$

2. Associativity

$$\forall a,b,c \in G, (a*b)*c = a*(b*c)$$

3. Unit element

$$\exists e \in G, \forall a \in G, e*a = a*e = a$$

   – Uniqueness: e = e' * e = e'.

4. Inverse element

$$\forall a \in G, \exists a^{-1} \in G, a*a^{-1} = a^{-1}*a = e$$

   – Uniqueness: $a'^{-1} = a'^{-1} * (a * a^{-1}) = (a'^{-1} * a) * a^{-1} = a^{-1}$.

Examples

Space inversion

$$\mathbf{r} \in R^3$$

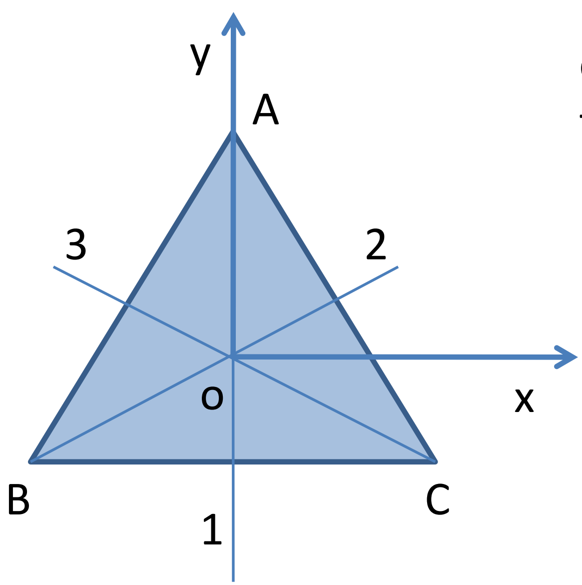
$$E\mathbf{r} = \mathbf{r}, I\mathbf{r} = -\mathbf{r}$$

$$G = \{E, I\}$$

	E	I
E	E	I
I	I	E

Multiplication table

D_3



D_3={e, d, f, a, b,c}

e: no operation

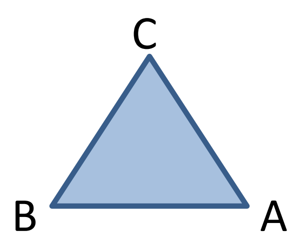
d: 2/3π about z axis

f: 4/3π about z axis

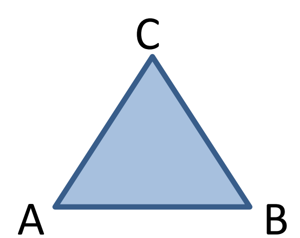
a: π about 1 axis

b: π about 2 axis

c: π about 3 axis



b



ab=d

Examples

Examples: Orthogonal and Unitary Groups

- Orthogonal n $\times$ n matrices form group O(n), and SO(n) if their determinants are +1(S stands for "special").

If $\tilde{O}_i = O_i^{-1}$ for i = 1 and 2 are elements of O(n), then the product

$O_1 O_2 = \tilde{O}_1 \tilde{O}_2 = O_2^{-1} O_1^{-1} = (O_1 O_2)^{-1}$ is also an orthogonal matrix in O(n), providing closure under (matrix) multiplication.

$$\begin{pmatrix} \cos\varphi_1 & \sin\varphi_1 \\ -\sin\varphi_1 & \cos\varphi_1 \end{pmatrix} \begin{pmatrix} \cos\varphi_2 & \sin\varphi_2 \\ -\sin\varphi_2 & \cos\varphi_2 \end{pmatrix} = \begin{pmatrix} \cos(\varphi_1+\varphi_2) & \sin(\varphi_1+\varphi_2) \\ -\sin(\varphi_1+\varphi_2) & \cos(\varphi_1+\varphi_2) \end{pmatrix}$$

- Likewise, unitary n $\times$ n matrices form the group U(n), and SU(n) if their determinants are +1. If $U_i^\dagger = U_i^{-1}$ are elements of U(n), then $(U_1U_2)^\dagger = U_2^\dagger U_1^\dagger = U_2^{-1} U_l^{-1} = (U_1U_2)^{-1}$, so the product is unitary and element of U(n), thus providing enclosure under multiplication. Each unitary matrix has an inverse , which again is unitary.

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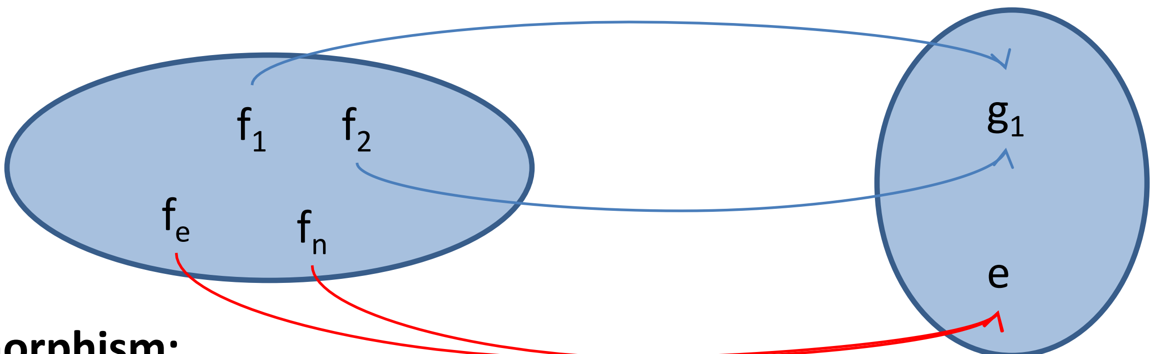
Mapping

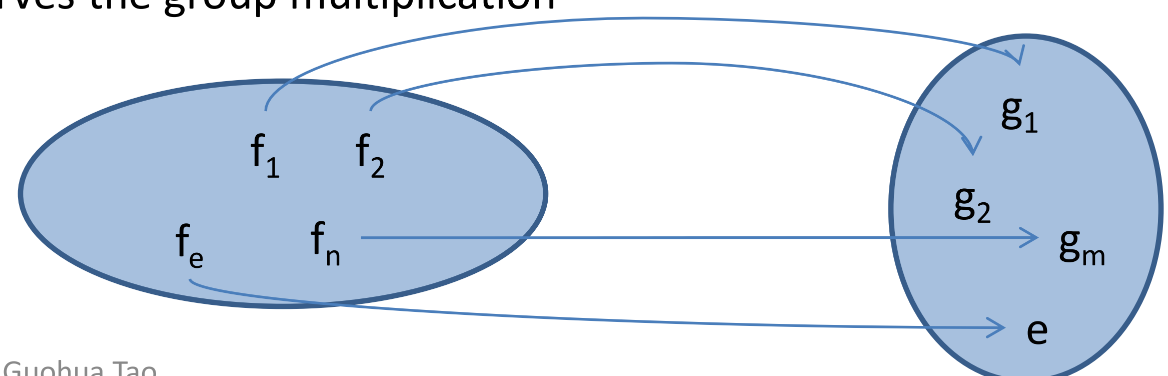
Homomorphism:

Correspondence between the elements of two groups (one-to-one, two-to – one, or many –to –one) that preserves the group multiplication



Isomorphism:

One-to-one correspondence between the elements of two groups that preserves the group multiplication



Examples

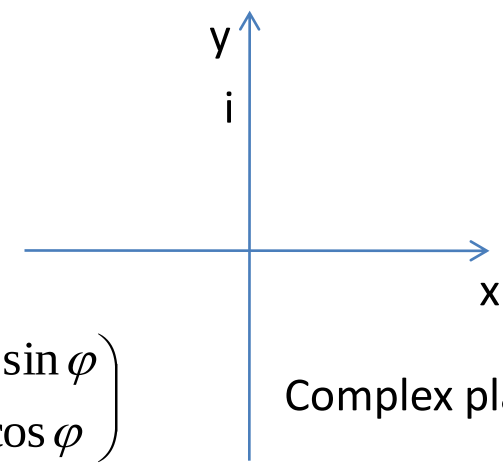
SO(3)

$$SO(3) = \{O \in O(3) \mid \det O = 1\}$$
$$O(3) = SO(3) \otimes \{E, I\}$$

Z_4

	1	a	b	c
1	1	a	b	c
a	a	b	C	1
b	B	c	1	a
c	c	1	a	b

$$1 \to 1, \quad a \to i,$$
$$b \to -1, \quad c -> -i$$



Complex plane

$$R(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \mathrm{C} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$Z_4 \leftrightarrow \{1, A, B, C\}$$

Vector space

The collection of vectors forms vector space. And it has the following properties:

1. Vector equality
   1. **A=B =>** $A_i = B_i$
2. Addition
   1. Associativity (**A**+**B**)+**C**=**A**+(**B+C**)
   2. Commutativity **A+B=B+A**
   3. Distributivity a*(**A+B**)=a***A**+a***B**; (a+b)***A**=a***A**+a***B**

   Commutative addition group

3. Scalar multiplication
   1. Compatibility a*(b***A**)=(a*b)***A**
   2. Identity element 1***A**=**A**
4. Negative of a vector (inverse element)

   A+(-**A**)=0
5. Null vector (identity element of addition)

   A+0=**A** for any **A**

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Linear mapping

Linear transformation:

$$\forall x, y \in V, a \in K(R\ or\ C)$$
$$A: V \to V, A(x) \in V$$
$$A(ax+y) = aA(x) + A(y)$$

$$(aA)(x) = a(A(x))$$
$$(A+B)(x) = A(x) + B(x)$$
$$(AB)(x) = A(B(x))$$

If A is a one-to-one mapping, $\exists A^{-1}$

N-dimensional linear space:

coordinates

Basis set:

$$(e_1, e_2, \ldots, e_n)$$
$$\forall x \in V, \exists x_i \in K$$
$$x = x_1 e_1 + x_2 e_2 + \ldots + x_n e_n = \sum_{i=1}^{n} x_i e_i$$

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ . \\ . \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ . \\ . \\ 0 \end{pmatrix}, \ldots, e_n = \begin{pmatrix} 0 \\ 0 \\ . \\ . \\ 1 \end{pmatrix}, \mathrm{x} = \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{pmatrix}$$

Representation

Linear transformation

$$\mathbf{A}=\begin{pmatrix} a_{11} & a_{12} & .. & .. & a_{1n} \\ a_{21} & a_{22} & .. & .. & a_{2n} \\ .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. \\ a_{n1} & a_{n2} & .. & .. & a_{nn} \end{pmatrix}$$

← Matrix

$$\mathbf{A}e_j=\sum_{i=1}^{n} a_{ij}e_i, (\mathbf{A}x)_i=\sum_{j=1}^{n} a_{ij}x_j$$

$$\text{If } \det \mathrm{A}\neq 0, \exists \mathrm{A}^{-1}.$$

Nonsingular

Group

If A is a homomorphic mapping from group G to the linear transformation group L(V,C) on linear space V, it is called a linear representation of group.

$$A: G\rightarrow L(V,C)$$

Linear transformation group

Matrix group

$$\forall g_a\in G, \exists A(g_a)\in L(V,C)$$

$$\forall g_a, g_b\in G, A(g_a g_b)=A(g_a)A(g_b)$$

$$A(g_0)=E_{n\times n}$$

$$A(g_a^{-1})=A^{-1}(g_b)$$

Examples

Reflection

$$\mathbf{A}(E)=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\quad \mathbf{A}(\sigma_k)=\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}$$

$V=R^3$

$C_k(\pi)$

$$B(E)=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\quad B(C_k(\pi))=\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix}$$

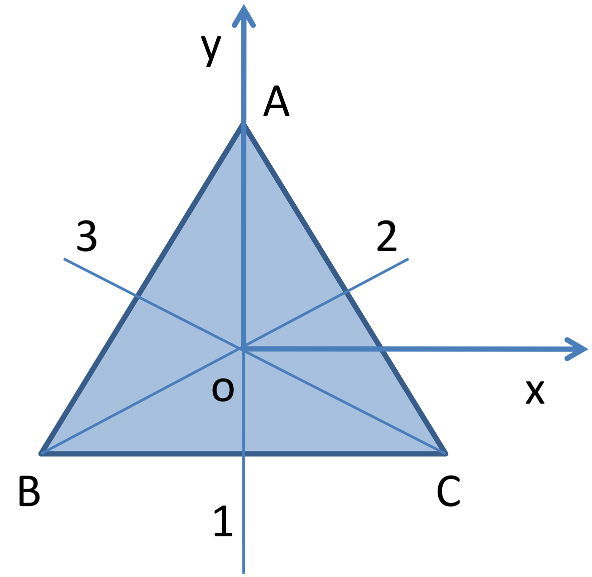
Inversion:

$$C(E)=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\quad C(I)=\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}$$

$$A(E)=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\quad A(I)=\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}$$

$$A(g_a)\varphi_i(\mathbf{r})=\varphi_i(g_a^{-1}\mathbf{r}), i=1,2,3\quad \mathrm{V}=\{\varphi_1,\varphi_2,\varphi_3\}$$

Examples



D_3 $V = Z_2$

$$D_3 = \{e, d, f, a, b, c\}; Z_2 = \{1, -1\}$$

$$\{e, d, f\} \to 1$$

$$\{a, b, c\} \to -1$$

$$A(e) = A(d) = A(f) = 1$$

$$A(a) = A(b) = A(c) = -1$$

$V = R^3$

$$A(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A(d) = \begin{pmatrix} -1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A(f) = \begin{pmatrix} -1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A(a) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad A(b) = \begin{pmatrix} 1/2 & \sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad A(c) = \begin{pmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$