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Chapter 3: Group Theory

1. Definition
2. Mapping
3. Representation
4. Continuous group
5. Lie group and Lie algebra
6. Discrete group

Reducible representation

Invariant subgroup:

If $H \subseteq G, \forall h \in H, \forall g \in G, ghg^{-1} \in H$

then H is an invariant subgroup of G

Reducible representation:

A is a representation of group *G* on the representation space *V*.

If $\exists W \subset V, W \neq \varnothing, \forall y \in W, \forall g \in G, A(g)y \in W,$

then A is a reducible representation.

Direct sum:

Linear space $V, V_1, V_2 \subseteq V, V_1 \cap V_2 = \varnothing$

if $\forall x \in V, \exists y \in V_1, \exists z \in V_2,$

$x = y + z$, or $V = V_1 + V_2$, then $V = V_1 \oplus V_2$.

Completely reducible representation:

A is a representation of group *G* on the representation space *V*.

$$V = V_1 \oplus V_2, \forall y \in V_1, \forall z \in V_2, \forall g \in G$$

$$\text{if } A(g)y \in V_1, A(g)z \in V_2$$

Representation

Reducible representation:

$$\exists(e_1, e_2, \ldots, e_m, e_{m+1}, \ldots, e_n) \in V$$
$$(e_1, e_2, \ldots, e_m, 0, \ldots, 0) \in W$$

$$A(g_\alpha) = \begin{pmatrix} B_m(g_\alpha) & D_m(g_\alpha) \\ 0 & C_n(g_\alpha) \end{pmatrix} \qquad y = \begin{pmatrix} Y_m \\ 0 \end{pmatrix} \in W, A(g_\alpha) y \in W$$

Completely reducible representation:

$$A(g_\alpha) = \begin{pmatrix} B_m(g_\alpha) & 0 \\ 0 & C_n(g_\alpha) \end{pmatrix} = B_m(g_\alpha) \oplus C_n(g_\alpha)$$

Irreducible representation:

A is a representation of group *G* on the representation space *V*.

~~If $\exists W \subset V, W \neq \varnothing, \forall y \in W, \forall g \in G, A(g) y \in W.$~~

Unitary representation

Inner product space:

Vector space V on number field K,

$\forall x, y \in V, \exists$ mapping $(x \mid y) \in K, \forall x, y, z \in V, a \in K$

(1) $(x+y \mid z)=(x \mid z)+(y \mid z)$; (2) $(x \mid ay)=a(x \mid y)$

(3) $(x \mid y)=(y \mid x)^{*}$; (4) $(x \mid x)>0$ if $x \neq 0$

Inner product $(x \mid y)$

Unitary transformation:

U a linear transformation on inner product space V,

$\forall x, y \in V, (Ux \mid Uy)=(x \mid y)$.

Unitary representation:

A is a representation of group *G* on the inner product space *V*,
A is a unitary transformation on *V*.

$$\forall g, h \in G,$$
$$A(gh)=A(g)A(h),$$
$$A(g)^{+}=A(g)^{-1}=A(g^{-1})$$

if $\exists(e_1, e_2, \ldots, e_m, e_{m+1}, \ldots, e_n) \in V$

[orthogonal and normalized]

$$A(g_\alpha)=\begin{pmatrix} B_m(g_\alpha) & 0 \\ 0 & C_n(g_\alpha) \end{pmatrix}$$

Continuous group

Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij} 1_2; \qquad \text{anticommutation}$$

$$\sigma_i \sigma_j = i\sigma_k; \qquad \text{cyclic permutation}$$

$$\boxed{\exp(i\sigma_k \theta) = 1_2 \cos\theta + i\sigma_k \sin\theta}$$

$$\exp(A) = \sum_{n=0}^{\infty} \frac{1}{n!} A^n; \quad \sin(A) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} A^{2n+1}; \quad \cos(A) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} A^{2n}$$

Rotation group SO(2):

$$\boxed{R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = 1_2 \cos\theta + i\sigma_2 \sin\theta = \exp(i\sigma_2 \theta)}$$

$$R(\theta_2)R(\theta_1) = \exp(i\sigma_2\theta_2)\exp(i\sigma_2\theta_1) = \exp(i\sigma_2(\theta_1+\theta_2)) = R(\theta_1+\theta_2)$$

Generator

Infinitesimal transformation:

$$R = \exp(i\varepsilon S), \varepsilon \to 0$$

If det(R)=1, $\det(R) = \exp\left(tr(\ln R)\right) = \exp\left(i\varepsilon tr(S)\right) = 1$

$$\Rightarrow tr(S) = 0$$

To determine the generator of rotation group SO(2):

$$-\left.\frac{idR(\theta)}{d\theta}\right|_{\theta=0} = -i\left.\begin{pmatrix} -\sin\theta & \cos\theta \\ -\cos\theta & -\sin\theta \end{pmatrix}\right|_{\theta=0} = -i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \sigma_2$$

The generators of rotation subgroups of SO(3)=?

$$R_x(\psi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi \end{pmatrix}; R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}; R_z(\varphi) = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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Lie group

Definition:

1. G is a group
2. G is a n-dimensional C^{∞} manifold
3. Both product ϕ and inverse τ are C^{∞} mapping

$$\phi : G \otimes G \to G; \qquad \tau : G \to G$$

$$\phi(\alpha, \beta) = \alpha\beta, \quad \tau(\alpha) = \alpha^{-1}, \quad \alpha, \beta \in \mathrm{G}$$

Reference:

Serge Lang, Real and functional analysis, 3rd edition, Springer-Verlag, New York, 1993.

Rotation group SO(2):

$$R_i = \exp(i\varepsilon_i S_i) = 1 + i\varepsilon_i S_i - \frac{1}{2}\varepsilon_i^2 S_i^2 + \ldots$$

$$R_i^{-1} = \exp(-i\varepsilon_i S_i) = 1 - i\varepsilon_i S_i - \frac{1}{2}\varepsilon_i^2 S_i^2 + \ldots$$

$$R_i^{-1} R_j^{-1} R_i R_j = 1 + \varepsilon_i \varepsilon_j \left[S_j, S_i\right] + \ldots$$

$$= 1 + \varepsilon_i \varepsilon_j \sum_k c_{ji}^k S_k + \ldots$$

Multiplication law of generators:

$$\left[S_i, S_j\right] = \sum_k c_{ij}^k S_k$$

Lie algebra

Definition:

V is n-dimensional vector space on number field R

$\forall X, Y \in V$, the Lie product $[X,Y] \in V$

1. Bilinear

$$[aX + bY, Z] = a[X, Z] + b[Y, Z]$$
$$[Z, aX + bY] = a[Z, X] + b[Z, Y]$$

2. Skew symmetric

$$[X, Y] = -[Y, X]$$

3. Jacobi identity

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$$

A vector space V with a bilinear skew-symmetric operation $V \times V \to V$ **, which satisfies the Jacobi identity.**

Example:

The set of n x n matrices becomes a Lie algebra if we define the commutator by [A,B]=AB-BA.

Discrete group

discrete group
continuous group

} Group {

finite group
infinite group

Examples:

- **Point group**: rotation and reflection (the origin is fixed).
  a finite subgroup of O(3).
  Space group: point group + translation
  32 distinct point groups and 230 space groups.
- SO(n)
- Additive group of integers

Theorem:

If group G is an nth-order group generated by rotations around a fixed axis k, then it is generated by $C_k(2\pi/n)$.

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Point group

- **Rotation C_n** (first type):

  N-th order cyclic group $\left\{C_n^1, C_n^2, \ldots, C_n^{n-1}, E\right\}$
- **Inverse** (second type):

  {E,I}
- **Rotation inverse**:

  IC_n

$$IC_{2n} \to \left\{C_n^1, C_n^2, \ldots, C_n^{n-1}, E, IC_{2n}, IC_{2n}^3, \ldots, IC_{2n}^{2n-1}\right\}$$

$$IC_{2n+1} \to \left\{C_{2n+1}^2, C_{2n+1}^4, \ldots, C_{2n+1}^{2n}, IC_{2n+1}, IC_{2n+1}^3, \ldots, IC_{2n+1}^{2n-1}, I\right\}$$

- **Reflection:**

  IC_2

$$\sigma_k = IC_k(\pi), \sigma_k^2 = E$$

The first type of point group:

The n_i order axis of group G C_{n_i}

Nonequivalent rotation n-1

$$\sum_{i=1}^{l} \frac{n}{2n_i}(n_i - 1) = n - 1; \quad \mathrm{n} \geq \mathrm{n_i} \geq 2$$

$$\Leftrightarrow \sum_{i=1}^{l} (1 - \frac{1}{n_i}) = 2(1 - \frac{1}{n})$$

Point group

1. l=2 $$\frac{2}{n}=\frac{1}{n_1}+\frac{1}{n_2}$$

   (1) C_n group: $n_1=n_2=n$, $n=2,3,\ldots$

2. l=3 $$1+\frac{2}{n}=\frac{1}{n_1}+\frac{1}{n_2}+\frac{1}{n_3}$$

   (2) D_m group: $n_1=n_2=2$, $n_3=m$ $n=2m$, $m=1,2,3,\ldots$
   (3) T group: $n_1=2$, $n_2=3$, $n_3=3$, $n=12$
   (4) O group: $n_1=2$, $n_2=3$, $n_3=4$, $n=24$
   (5) Y group: $n_1=2$, $n_2=3$, $n_3=5$, $n=60$