$$p_0^{est}(q) = \sum_{i=1}^{n} c_i(q) \frac{Z_i}{Z_0} e^{\beta w_i(q)} p_i^{est}(q)$$

$$Var(p_0^{est}(q)\Delta q)$$

$$= \sum_{i=1}^{n} c_i^2(q) \cdot \frac{Z_i^2}{Z_0^2} e^{2\beta w_i(q)} \quad Var(p_i^{est}(q)\Delta q)$$

$$\sum_{i=1}^{n} c_i(q) = 1 \qquad \frac{p_i(q)\Delta q}{M_i}$$

$$Var(p_0^{est}(q)\Delta q) \quad = \frac{Z_i}{Z_0} e^{\beta w_i(q)} \cdot p_i(q)$$

$$= p_0(q) \sum_{i=1}^{n} c_i^2(q) \cdot \frac{Z_i}{Z_0} e^{\beta w_i(q)} \cdot \frac{\Delta q}{M_i}$$

$$\frac{\partial Var(p_0^{est}(q)\Delta q)}{\partial c_i(q)} - \alpha \cdot \frac{\partial \sum_i c_i(q)}{\partial c_i(q)} = 0$$

$$2c_i(q) \cdot \frac{Z_i}{Z_0} e^{\beta w_i(q)} \Delta q / M_i = \alpha$$

$$\underline{C_i(q)} = \frac{\alpha}{2} \cdot \frac{Z_0}{Z_i} e^{-\beta W_i(q)} \cdot \frac{M_i}{\Delta q}$$

$$\alpha' = \frac{\alpha}{2\Delta q}$$

$$= \frac{\alpha' \cdot \frac{Z_0}{Z_i} e^{-\beta W_i(q)} \cdot M_i}{}$$

$$\alpha' = \frac{1}{\sum_{j=1}^{n} e^{-\beta W_j(q)} \cdot \frac{Z_0}{Z_j} \cdot M_j}$$

$$C_i(q) = \frac{\frac{Z_0}{Z_i} e^{-\beta W_i(q)} \cdot M_i}{\sum_{j=1}^{n} e^{-\beta W_j(q)} \cdot \frac{Z_0}{Z_j} \cdot M_j}$$

$$\underline{p_0^{est}(q)}\Delta q \,\underline{=}\, \sum_{i=1}^{n} \underline{C_i(q)} \cdot \frac{Z_i}{Z_0} e^{\beta W_i(q)} \, \frac{H_i(q)}{M_i}$$

$$= \sum_{i=1}^{n} \frac{\cancel{Z_0}/Z_i \, e^{-\beta \cancel{W_i(q)}} \cdot \cancel{M_i} \cdot \frac{\cancel{Z_i}}{\cancel{Z_0}} e^{\cancel{\beta W_i(q)}} \cdot \frac{H_i(q)}{\cancel{M_i}}}{\sum_{k=1}^{n} e^{-\beta W_k(q)} \cdot \frac{Z_0}{Z_k} \cdot M_k}$$

i_{th} state simulation:

of $[q, q+\Delta q]$

$$= \frac{\sum_{i=1}^{n} \underline{H_i(q)}}{\sum_{k=1}^{n} e^{-\beta W_k(q)} \cdot \frac{Z_0}{Z_k} \cdot \underline{M_k}}$$

$$\underline{Z_i} = Z_i \cdot 1 = Z_i \cdot \int dq \, P_i(q)$$

$$= \int dq \cdot P_i(q) \cdot Z_i$$

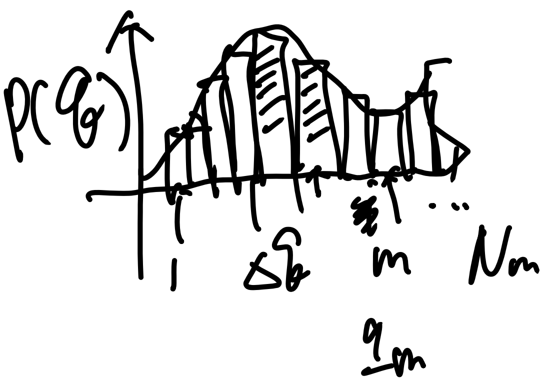
$$= \int dq \, \frac{Z_0}{\cancel{Z_i}} \cdot e^{-\beta W_i(q)} \cdot P_0(q) \cdot \cancel{Z_i}$$

$$\frac{Z_i}{Z_0} = \underbrace{\int dq \, e^{-\beta W_i(q)} \cdot P_0(q)}_{} = Z_i'$$

$$Z_i' \approx \int dq\, e^{-\beta W_i(q)} \cdot \underbrace{p_0^{est}(q)}$$

$$= \int dq \cdot e^{-\beta W_i(q)} \frac{1}{\Delta q} \cdot \frac{\sum_{j=1}^{n} H_j(q)}{\sum_{k=1}^{n} e^{-\beta W_k(q)} \frac{Z_i'}{Z_k} M_k}$$

$$= \int dq \cdot e^{-\beta W_i(q)} \cdot \frac{1}{\Delta q} \cdot \frac{\sum_{j=1}^{n} H_j(q)}{\sum_{k=1}^{n} e^{-\beta W_k(q)} \frac{M_k}{Z_k'}}$$



$$= \boxed{\sum_{m=1}^{N_m} e^{-\beta W_i(q_m)} \cdot \Delta q \cdot \frac{1}{\Delta q} \times \frac{\sum_{j=1}^{n} H_j(q_m)}{\sum_{k=1}^{n} e^{-\beta W_k(q)} \frac{M_k}{Z_k'}}}$$

WHAM

$$\underline{Z_i'} \approx \sum_{m=1}^{N_m} \frac{e^{-\beta W_i(q_m)} \cdot \sum_{j=1}^{n} H_j(q_m)}{\sum_{k=1}^{n} e^{-\beta W_k(q_m)} \cdot \frac{M_k}{\underline{Z_k'}}} \quad \text{(A)}$$

$$Z_i' = 1 \xrightarrow{\text{r.h.s (A)}} \underbrace{Z_i'^{(1)}}_{\text{l.h.s}} \xrightarrow{\text{r.h.s (A)}} Z_i'^{(2)} \rightarrow \cdots \quad n$$

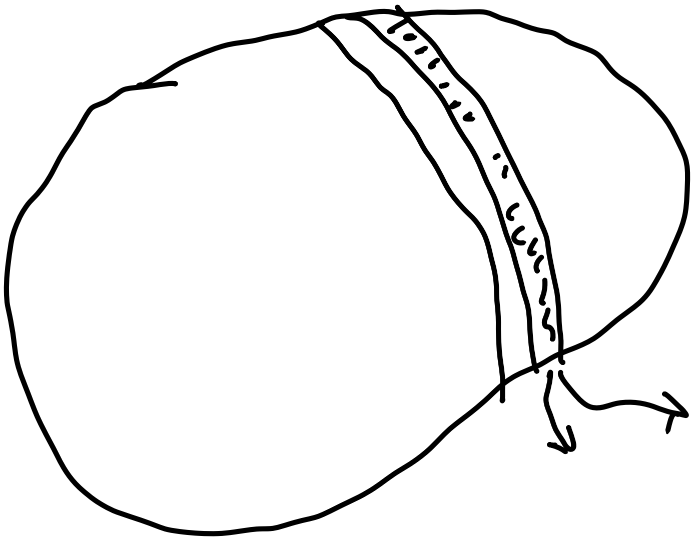
$$Z_i'^{(N)} = \underline{Z_i'^{(N+1)}}$$

(B)

$$\underline{P_0^{est}(q)\Delta q} = \frac{\sum_{j=1}^{n} H_j(q)}{\sum_{k=1}^{n} e^{-\beta W_k(q)} \cdot \frac{M_k}{Z_k'}}$$

$$A(\vec{q}_N) \rightarrow \bar{A}\ ?$$

$$\bar{A} = \frac{1}{Z} \int d q_N \cdot e^{-\beta U_0(q_N)} \cdot A(q_N)$$



configurational space.

$$P_0(q)\,dq = \frac{1}{Z}\int dq_N\, e^{-\beta U_0(q_N)} \times \delta(q(q_N) - q)$$

$$\bar{A} = \int dq \cdot P_0(q) \cdot \underline{A_q}$$

$$= \frac{1}{Z}\int dq \cdot A_q \cdot \int dq_N\, e^{-\beta U_0(q_N)} \times \delta(q(q_N) - q)$$

$$= \frac{1}{Z}\int dq \cdot \int dq_N \cdot e^{-\beta U_0(q_N)}\, \delta(q(q_N) - q)\, A(q_N)$$

$$= \frac{1}{Z}\int dq \cdot \frac{\underline{\int dq_N\, e^{-\beta U_0(q_N)} \times \delta(q(q_N) - q)}}{\int dq_N\, e^{-\beta U_0(q_N)} \times \delta(q(q_N) - q)} \times$$

$$\int dq_N \, e^{-\beta U_0(q_N)} \times \delta(\xi(q_N) - \xi) \times A(q_N)$$

$$= \int d\xi \cdot P_0(\xi) \cdot \frac{\int dq_N \, e^{-\beta U_0(q_N)} \times \delta(\xi(q_N) - \xi) \times A(q_N)}{\int dq_N \, e^{-\beta U_0(q_N)} \times \delta(\xi(q_N) - \xi)}$$

$$A_\xi = \frac{\int dq_N \, e^{-\beta U_0(q_N)} \times \delta(\xi(q_N) - \xi) \times A(q_N)}{\int dq_N \, e^{-\beta U_0(q_N)} \times \delta(\xi(q_N) - \xi)}$$

$$A_\xi^{est} = \frac{\sum_{j=1}^{n} \sum_{l=1}^{M_j} A_{jl} \cdot I\{\xi_{jl} \in [\xi, \xi + \Delta\xi]\}}{\sum_{j=1}^{n} H_j(\xi)}$$

C

Conditional average.

$$\bar{A} = \sum P_0^{est}(\xi) \, A_\xi^{est}$$

q_a, q_b

$P_0(q_a, q_b)$

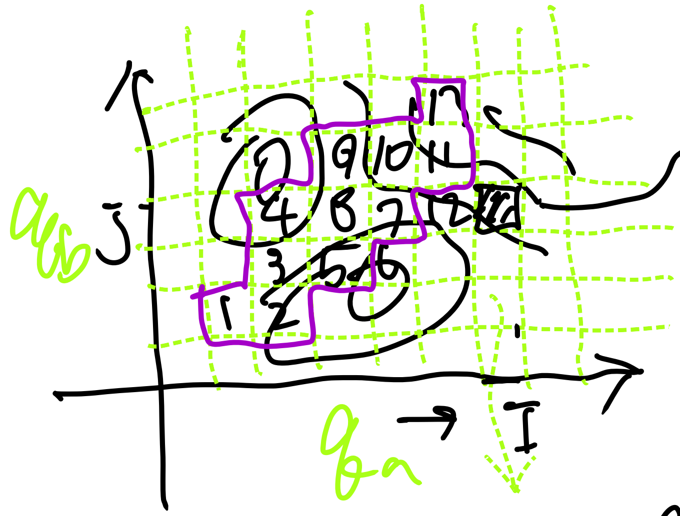
$$\underbrace{V_{ij}}_{l} = V_0(q_n) + W_i(q_a(q_n)) + V_j(q_b(q_n))$$

l	i	j
1	1	1
2	1	2
3	1	3
4	1	4
⋮		⋮
11	2	1
12	2	2
13	⋮	⋮
⋮		

$$\Rightarrow V_l = V_0(q_n) + W_l(q_a(q_n)) + V_l(q_b(q_n))$$

$l = 1$, $W_l = W_1$, $V_l = V_1$

$l = 11$, $W_l = W_2$, $V_l = V_1$



$H(q_{aI}, q_{bJ})$

$H(q_{am}, q_{bm})$

$[(q_a, q_b), (q_a + \Delta q_a, q_b + \Delta q_b)]$

m=1

m=2.

m	I	J
1	1	1
2	1	2
3	1	3
⋮		⋮

$q_{am} = q_{aI}$

$q_{bm} = q_{aJ}$

$W_\ell = w_\ell + V_\ell$

$$Z'_\ell = \sum_{m=1}^{N_m} e^{-\beta W_\ell(q_{am}, q_{bm})} \frac{\sum_{j=1}^{N_\ell} H_\ell(q_{am}, q_{bm})}{\sum_{k=1}^{N_\ell} e^{-\beta W_k(q_{am}, q_{bm})} \cdot \frac{N_k}{Z'_k}}$$

① $U_i = U_0 + w_i$

② $\underline{\underline{\beta}}, P, \rho, \mu$

Let states defined with different β_i

obtain. $P_i(q_\theta) \longrightarrow P_0(q_\theta)$

$$P_i(q_\theta) = \langle \delta(q_\theta(q_N) - q_\theta) \rangle_{\beta_i}$$

$$= \frac{1}{Z_i} \int_{dq_N} e^{-\underbrace{\beta_i U_0(q_N)}_{x}} \delta(q_\theta(q_N) - q_\theta)$$

$$P_i(q_\theta, U) = \langle \delta(q_\theta(q_N) - q_\theta)\delta(U_0(q_N) - U) \rangle_{\beta_i}$$

$$= \frac{1}{Z_i} \int_{dq_N} e^{-\beta_i U(q_N)} \delta(q_\theta(q_N) - q_\theta) \times \underline{\delta(U_0(q_N) - U)}$$

$$= \frac{1}{Z_i} e^{-\beta_i U} \underbrace{\int dq_N \, \delta(q_\theta(q_N) - q_\theta)\delta(U_0(q_N) - U)}$$

does not depend β

$$= \frac{1}{Z_i} e^{-\beta_i U} \underbrace{\Omega(q, \underline{U})}$$ → reduced configurational density.

$$\Omega(N, \underline{E}, V)$$

$$P_0(q, U) = \frac{1}{Z_0} e^{-\beta U} \cdot \underline{\Omega(q, U)}$$

$$P_0(q, U) = \frac{Z_i}{Z_0} e^{(\beta_i - \beta) U} \cdot P_i(q, U)$$

$$P_0(q) = \frac{Z_i}{Z_0} e^{\beta W_i(q)} \cdot P_i(q, U)$$

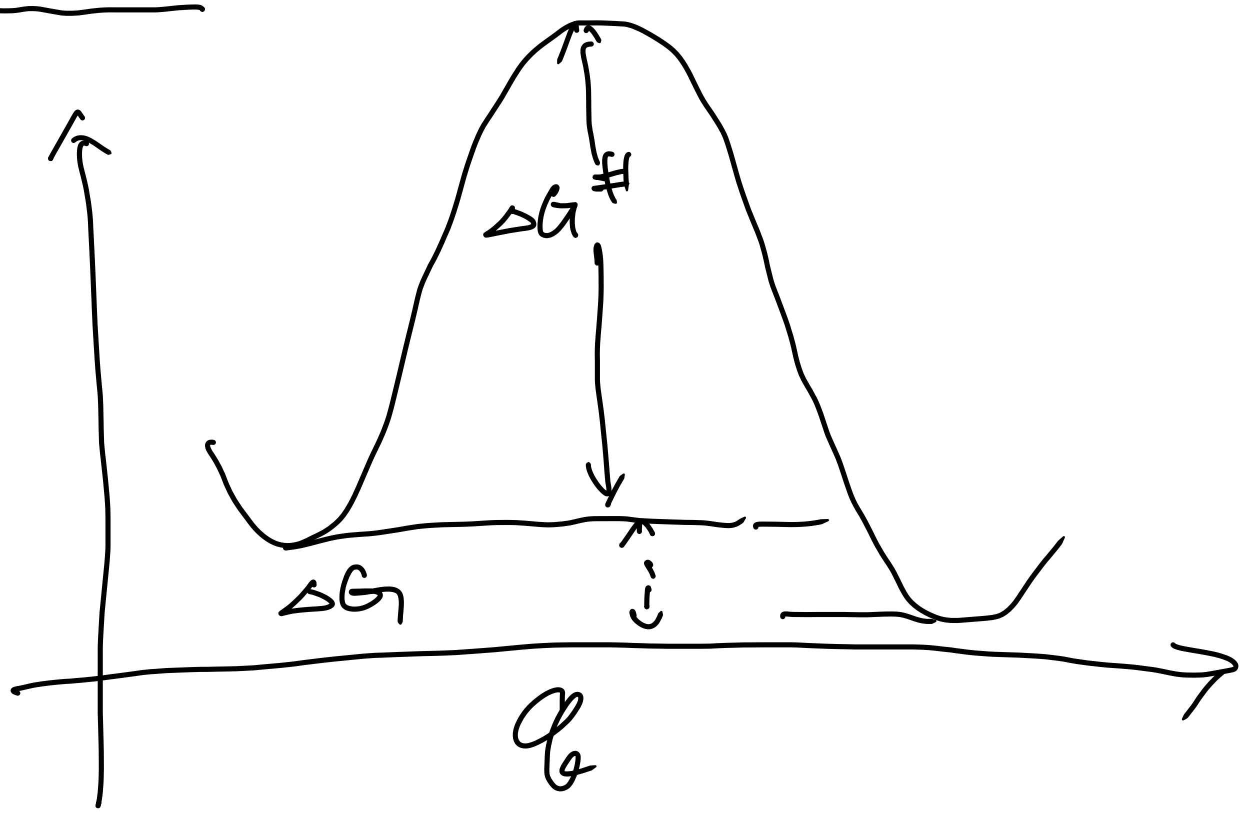
$$W_i(q) = \frac{\beta_i - \beta}{\beta} U_0(q_n)$$

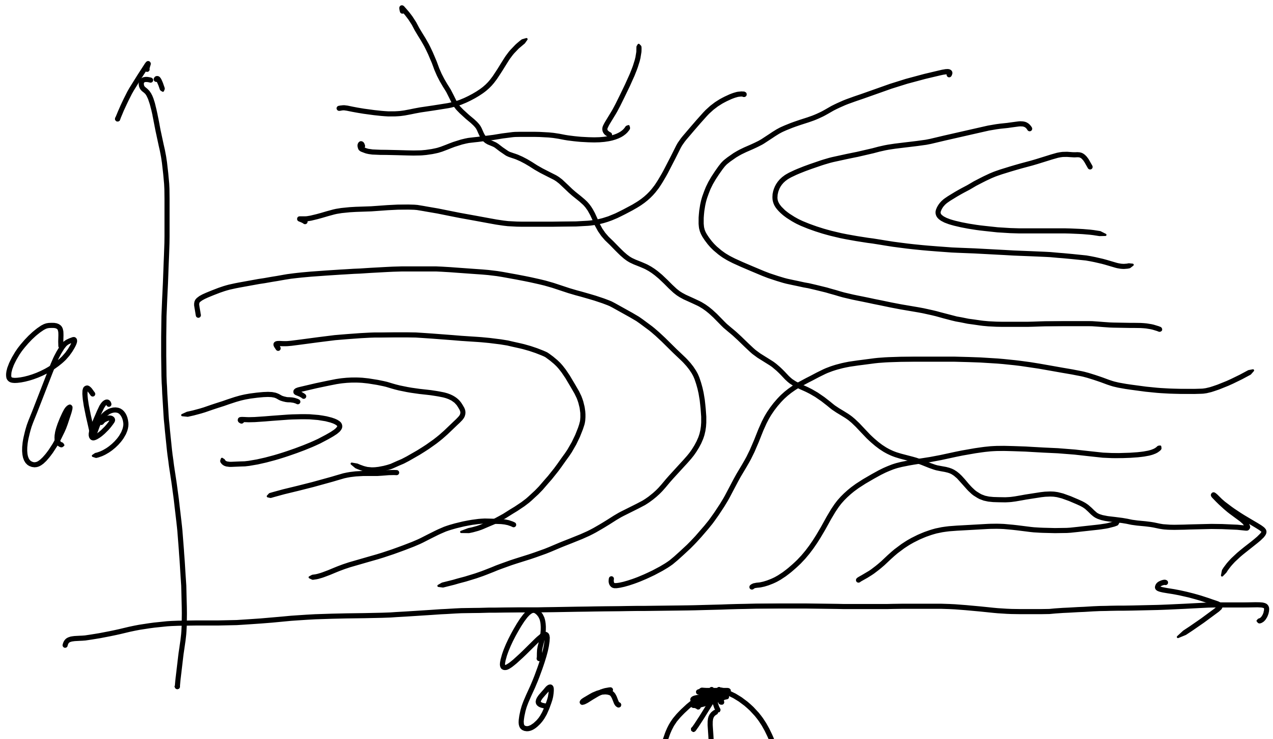
$$U_i = U_0(q_n) + \underbrace{\left(\frac{\beta_i - \beta}{\beta}\right) U_0(q_n)}_{W_i}$$

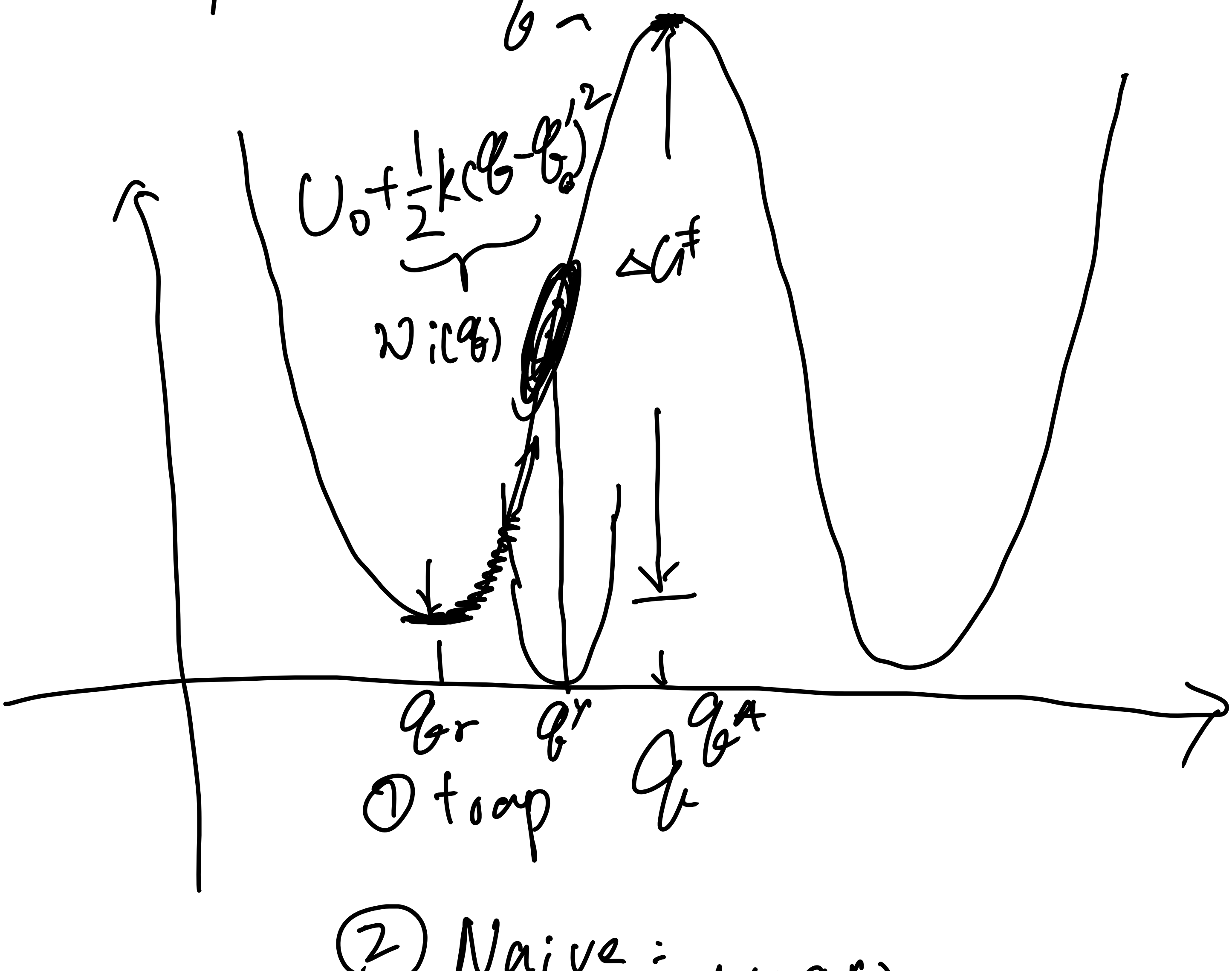
$$Z_l' = \sum_{m=1}^{N_m} e^{-(\beta_l-\beta)U(q_m)} \frac{\sum_{j=1}^{N_e} H_j(q_m, U_m)}{\sum_{k=1}^{N_e} e^{-(\beta_k-\beta)U(q_m)} \cdot \frac{M_k}{Z_k'}}$$

$$P_0^{est}(q_\theta, U)\Delta q_\theta \Delta U = \frac{\sum_{j=1}^{N_e} H_j(q_\theta, U)}{\sum_{k=1}^{N_e} e^{-(\beta_k-\beta)U(q_\theta)} \cdot \frac{M_k}{Z_k'}}$$

$$\underline{P_0^{est}(q_\theta)} = \sum_{i=1}^{N_U} P_0^{est}(q_\theta, U_i)$$





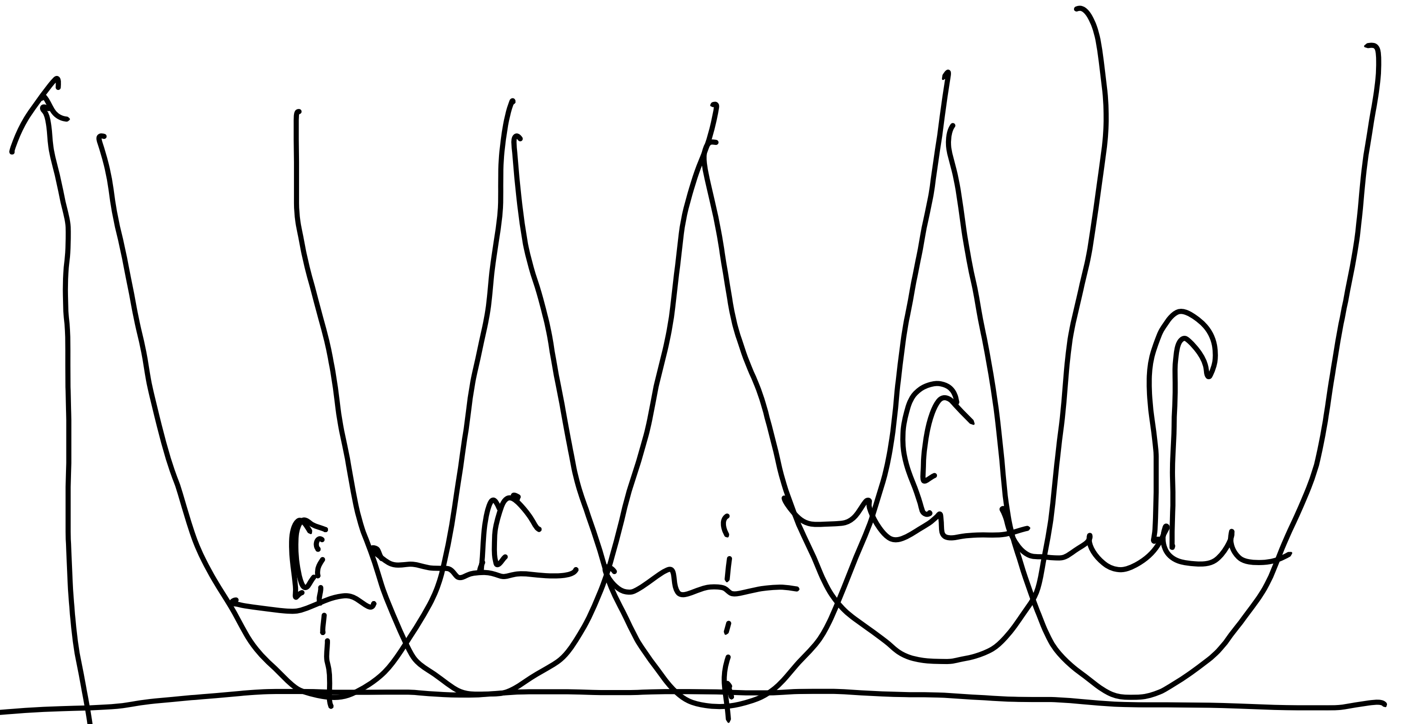


② Naive:

$$-kT \ln \frac{N(q^*)}{N(q_r)} \sim \Delta G^\ddagger$$

$\hookrightarrow 10^6$

$$-kT\ln\frac{\underline{N(q^*)}+\underline{\delta N}}{N(q_r)+\underline{\underline{\delta N}}}$$



$w_1 = \frac{1}{2}k_1(q-q_{10})^2$ $\quad q \quad$ $w_i = \frac{1}{2}k_i(q-q_{i0})^2$

$$U_i = U_0 + w_i = U_0 + \frac{1}{2}k_i(q-q_{i0})^2$$

umbrella sampling

$P_0(q)$

# of US windows

Number of q bin

$$Z_i' = \sum_{m=1}^{N_m} e^{-\frac{1}{2}\beta k_i (q_m - q_{i0})^2} \times \frac{\sum_{j=1}^{n} H_j(q_m)}{\sum_{k=1}^{n} e^{-\frac{1}{2}\beta k_k (q_m - q_{k0})^2} \cdot \frac{N_k}{Z_k'}}$$

N_m, n are independent of each other

US

① setup n simulation

i: $U_0 + \frac{1}{2} k_i (q - q_{i0})^2$

most case, $q_{i0} = q_{a0} + i \times \underbrace{\Delta q_0}$

distance between neighboring umbrella potential

② collect n trjs, compile all q data, do the histogram. with N_m bin,

bin width is thicner than Δq_0

③ WHAM. $\rightarrow P_0^{est}(q)$