Note

- 11月17日23:59之前必须提交口试讲稿，过期未交者扣除口试部分材料得分（25分）。
- 12月5日、12月12日口试（总分50分），12月19日复习课，期末考试（开卷，总分50分）时间待定。
- 平时作业可自行选做参考书中习题（不必提交，不记成绩）。

  参考书：Mathematical Methods for Physicists, George B. Arfken and Hans J. Weber, Academic Press.

Chapter 4: Functions of a complex variables

1. Algebra
2. Mapping
3. Analytic functions
4. Cauchy integral theorem
5. Residue theorem
6. Examples

Algebra

Complex variables: an ordered pair of two real variables z=(x, y)

1. Addition

$$z_1 + z_2 = (x_1, y_1) + (x_2, y_2)$$

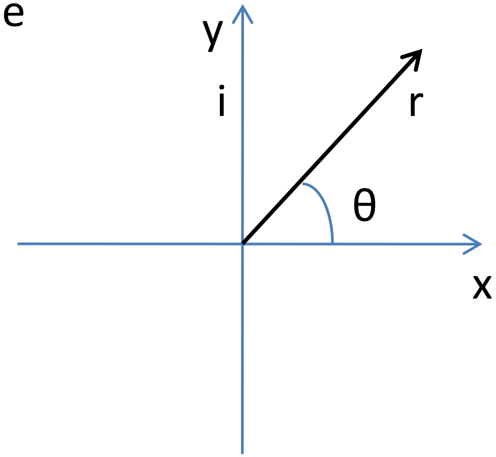
2. Multiplication

$$z_1 z_2 = (x_1, y_1)\cdot(x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$$

$$z = x + iy, i^2 = -1 \qquad z^* = x - iy, zz^* = |z|^2$$

magnitude

3. The polar form



$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

$$|z_1| - |z_2| \le |z_1 + z_2| \le |z_1| + |z_2|$$

$$|z_1 z_2| = |z_1|\cdot|z_2|, \arg(z_1 z_2) = \arg z_1 + \arg z_2$$

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Mapping

Functions of a Complex Variables

$$w = f(z) = u(x, y) + iv(x, y)$$

$$z - \text{plane}\,(x, y) \xrightarrow{f} w - \text{plane}\,(u, v)$$

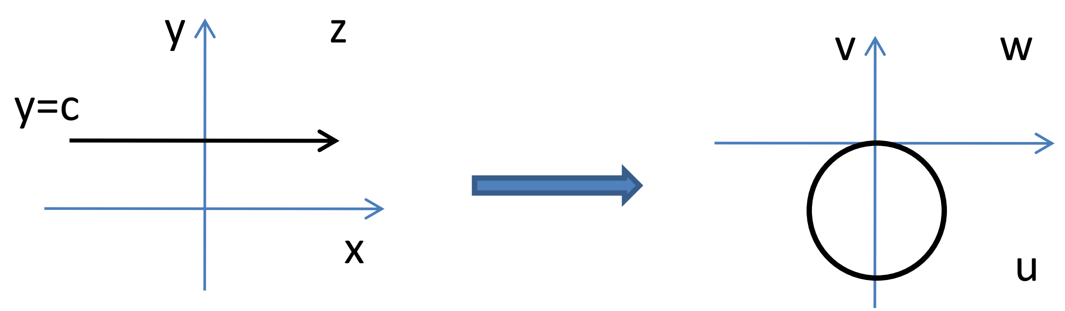
1. Translation

$$w = z + z_0$$

2. Rotation

$$w = zz_0$$

3. Inversion



$$w = \frac{1}{z} = \rho e^{i\varphi}, z = re^{i\theta}$$

$$\Rightarrow \rho = \frac{1}{r}, \varphi = -\theta$$

$$u + iv = \frac{1}{x + iy} \Rightarrow u = \frac{x}{x^2 + y^2}, v = -\frac{y}{x^2 + y^2}$$

$$x = \frac{u}{u^2 + v^2}, y = -\frac{v}{u^2 + v^2}$$

$$y = c \Rightarrow u^2 + \left(v + \frac{1}{2c}\right)^2 = \left(\frac{1}{2c}\right)^2$$

Multivalent functions

Functions of a Complex Variables

$$w = \rho e^{i\varphi}, z = re^{i\theta}$$

Cut line

Two-to-one

- $w = z^2 \Rightarrow \rho = r^2, \varphi = 2\theta$

$$u + iv = (x + iy)^2 \Rightarrow u = x^2 - y^2, v = 2xy$$

One-to-two

- $w = z^{1/2} \Rightarrow \rho = r^{1/2}, 2\varphi = \theta$

Riemann surface of $z^{1/2}$

Many-to-one

- $w = e^z \Rightarrow \rho = e^x, \varphi = y$

Images here are removed due to copy right issues

One-to-many

- $w = \ln z \Rightarrow u = \ln r, v = \theta + 2n\pi$

Riemann surface of lnz

Analytic functions

Derivative of a general function:

$$\lim_{\delta z\to 0}\frac{f(z+\delta z)-f(z)}{z+\delta z-z}=\lim_{\delta z\to 0}\frac{\delta f(z)}{\delta z}=f'(z)$$

1. Derivative of a real function:

$$z=x;\delta z=\delta x$$

$$\lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{x+\delta x-x}=\lim_{\delta x\to 0}\frac{\delta f(x)}{\delta x}=f'(x)$$

2. Derivative of a complex function:

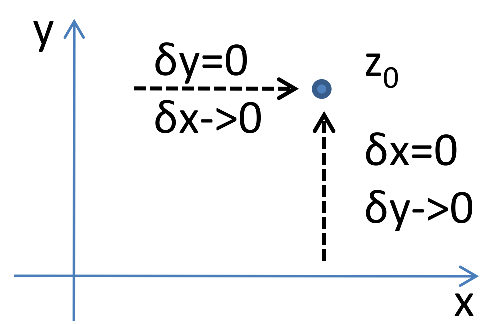
$$\left.\begin{aligned}\delta z&=\delta x+i\delta y\\ \delta f&=\delta u+i\delta v\end{aligned}\right\}\Longrightarrow \frac{\delta f}{\delta z}=\frac{\delta u+i\delta v}{\delta x+i\delta y}$$

$$\lim_{\delta z\to 0}\frac{\delta f(z)}{\delta z}\overset{\delta y=0}{=}\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

$$\lim_{\delta z\to 0}\frac{\delta f(z)}{\delta z}\overset{\delta x=0}{=}-i\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}$$

3. Cauchy-Riemann conditions

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y};\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$



Analytic functions

1. Analytic:

Cauchy-Riemann conditions $\Longleftrightarrow$ $\exists f'(z)$

$$\exists V(z_0;\varepsilon)=\{z:|z-z_0|<\varepsilon\}, \forall z\in V$$

if $f(z)$ is differentiable, then $f(z)$ is analytic at z_0

If $f(z)$ is analytic in field D, then $f(z)$ is an entire function.

2. Conformal mapping:

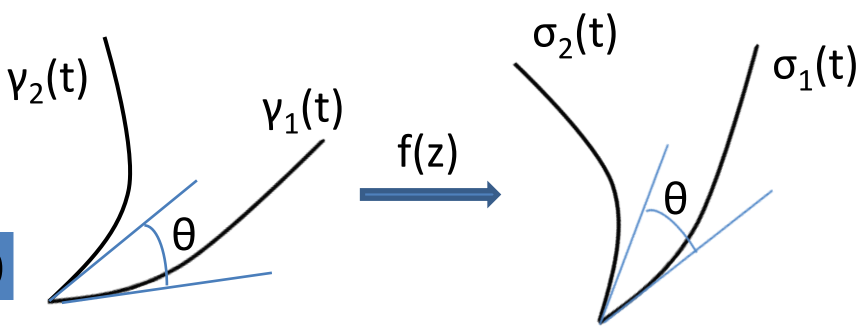
$\gamma(t)$ smooth curve in field $D: 0\le t\le 1, \gamma(0)=z_0$

$$\sigma(t)=f[\gamma(t)], \sigma'(t)=f'[\gamma(t)]\gamma'(t)$$

$$\sigma'(0)=f'(z_0)\gamma'(0)$$

$$\arg\sigma'(0)=\arg f'(z_0)+\arg\gamma'(0)$$

$$\arg\sigma_2'(0)-\arg\sigma_1'(0)=\arg\gamma_2'(0)-\arg\gamma_1'(0)$$



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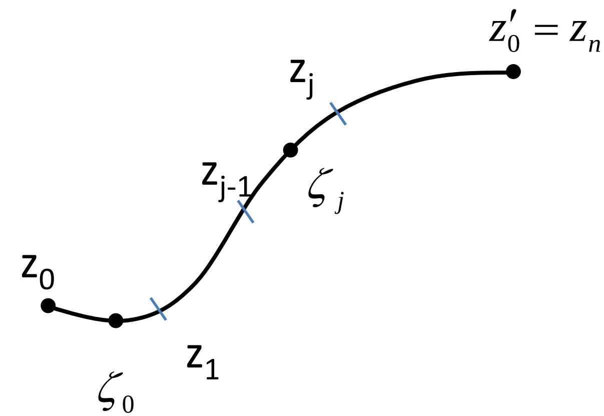
Contour integral

If the limit exists,

$$S_n = \sum_{j=1}^{n} f(\zeta_i)(z_j - z_{j-1})$$

$$\int_{z_0}^{z_0'} f(z)dz = \lim_{n\to\infty} \sum_{j=1}^{n} f(\zeta_i)(z_j - z_{j-1})$$



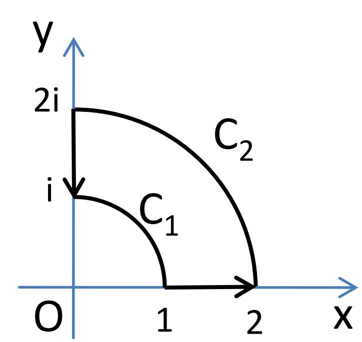
Examples:

$$(1)\ \ I = \frac{1}{2\pi i} \int_{C:|z|=r>0} z^n = ?$$

$$I = \frac{ir^{n+1}}{2\pi i} \int_0^{2\pi} \exp[i(n+1)\theta] d\theta = 0$$

$$(n \neq -1)$$

$$I = 1 \quad (n = -1)$$



$$(2)\ \ I = \int_C \frac{dz}{\bar{z}} = ?$$

$$I = \int_1^2 \frac{dx}{x} + \int_2^1 \frac{idy}{-iy} + \int_{C_1} \frac{zdz}{|z|^2} + \int_{C_2} \frac{zdz}{|z|^2} = 2\log 2$$

Cauchy integral theorem

<u>Stokes's theorem proof:</u>

a simply connected region R, a function f(z) is analytic and its first partial derivatives are continuous, for any closed path C in R,

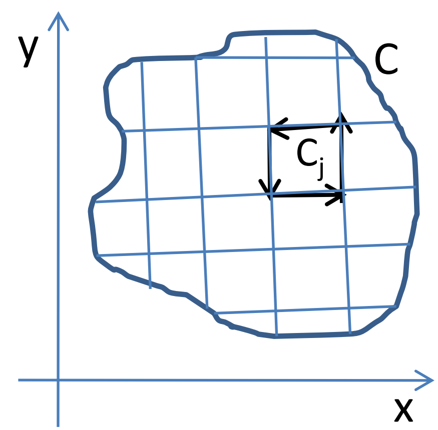
$$\oint_C f(z)dz = 0?$$

$$\oint_C f(z)dz = \oint_C udx - vdy + i\oint_C vdx + udy$$

$$= \int_R \left(\frac{\partial(-v)}{\partial x} - \frac{\partial u}{\partial y} \right) dxdy + i\int_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dxdy = 0$$

<u>Cauchy-Goursat proof:</u>

first partial derivatives are finite



$$\oint_C f(z)dz = \sum_j \oint_{C_j} f(z)dz = 0?$$

$$\delta_j(z, z_j) = \frac{f(z) - f(z_j)}{z - z_j} - f'(z_j), \text{then} \left|\delta_j(z - z_j)\right| < \varepsilon$$

$$\oint_{C_j} f(z)dz = \underbrace{\oint_{C_j} f(z_j)dz}_{0} + \underbrace{\oint_{C_j} f'(z_j)(z - z_j)dz}_{0} + \underbrace{\oint_{C_j} \delta(z, z_j)(z - z_j)dz}_{0}$$

Cauchy integral formula

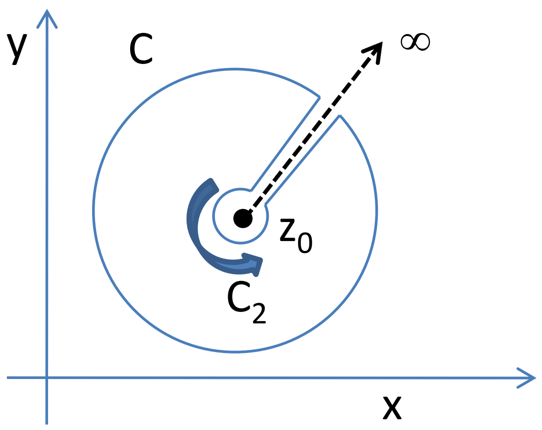
z_0 is some point in the interior region bounded by C.

$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}dz = f(z_0)$$

$f(z)/(z-z_0)$ is not analytic at $z=z_0$.

$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}dz - \frac{1}{2\pi i}\oint_{C_2} \frac{f(z)}{z-z_0}dz = 0$$

$C_2 : |z-z_0| = r.$



$$\frac{1}{2\pi i}\oint_{C_2} \frac{f(z)}{z-z_0}dz = \frac{1}{2\pi i}\oint_{C_2} \frac{f(z_0+re^{i\theta})}{re^{i\theta}} rie^{i\theta}d\theta \xrightarrow{r\to 0} \frac{1}{2\pi}\oint_{C_2} f(z_0)d\theta = f(z_0)$$

General formula:

$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}dz = \begin{cases} f(z_0), z_0 \text{ interior} \\ 0, \qquad z_0 \text{ exterior} \end{cases}$$

Derivative:

$$f^n(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz$$

Laurent expansion

Taylor expansion:

$$f(z)=\frac{1}{2\pi i}\oint_C \frac{f(z')dz'}{z'-z}=\frac{1}{2\pi i}\oint_C \frac{f(z')dz'}{z'-z_0}\frac{1}{1-t} \qquad t=\frac{z-z_0}{z'-z_0}$$

$$=\frac{1}{2\pi i}\oint_C \sum_{n=0}^{\infty}\frac{(z-z_0)^n f(z')dz'}{(z'-z_0)^{n+1}}$$

Uniformly convergent for |t|<1

$$=\sum_{n=0}^{\infty}(z-z_0)^n\frac{f^n(z_0)}{n!}$$

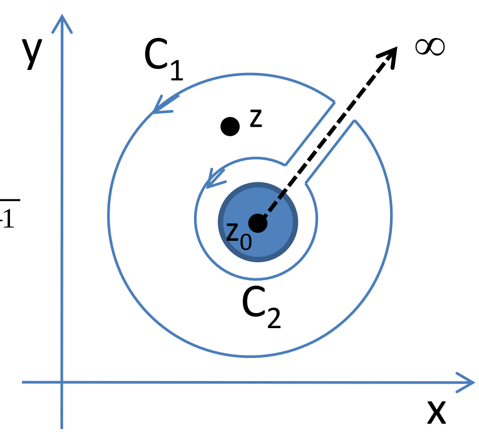
Laurent expansion:

$$f(z)=\frac{1}{2\pi i}\oint_{C_1}\frac{f(z')dz'}{z'-z}-\frac{1}{2\pi i}\oint_{C_2}\frac{f(z')dz'}{z'-z_0}$$

$$=\frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n\oint_{C_1}\frac{f(z')dz'}{(z'-z_0)^{n+1}}+\frac{1}{2\pi i}\sum_{n=1}^{\infty}(z-z_0)^{-n}\oint_{C_1}\frac{f(z')dz'}{(z'-z_0)^{n-1}}$$

$$=\sum_{n=-\infty}^{\infty}a_n(z-z_0)^n$$

ex. $$\frac{1}{z(z-1)}=-\sum_{n=-1}^{\infty}z^n$$



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Singularity

Isolated singular point: f(z) is analytic in $V^*(z_0; R)$.

Laurent expansion: $$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

Poles: z_0 is a pole of order m, $a_n = 0$ (n < -m < 0) and $a_{-m} \neq 0$

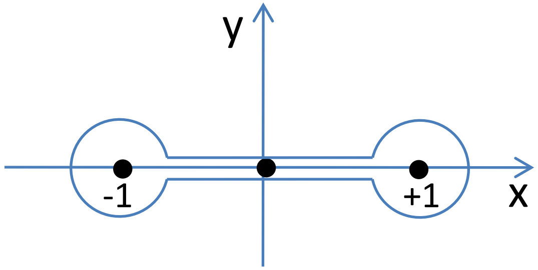
Essential singularity: z_0 is a pole of order ∞ $\quad \sin z; \quad e^{1/z}$

Branch points:

z_0 is a branch point, if f(z) is a multivalued function as z moves 2π around z_0

The cut line connects the branch points

ex. $$f(z) = (z^2 - 1)^{1/2} = (z+1)^{1/2}(z-1)^{1/2}$$



Residue theorem

Laurent expansion:

$$f(z)=\sum_{n=-\infty}^{\infty}a_n(z-z_0)^n \qquad a_n=\frac{1}{2\pi i}\oint_{C_1}\frac{f(z')dz'}{(z'-z_0)^{n+1}}$$

Integrated term by term:

$$a_n\oint_C (z-z_0)^n dz = a_n \left.\frac{(z-z_0)^{n+1}}{n+1}\right|_{z_1}^{z_1} = 0,\quad n\neq -1$$

$$a_{-1}\oint_C (z-z_0)^{-1} dz = a_{-1}\oint_C \frac{ire^{i\theta}d\theta}{re^{i\theta}} = 2\pi i a_{-1},\quad n\neq -1$$

$$\therefore \frac{1}{2\pi i}\oint_C f(z)dz = a_{-1}$$

The case of a set of isolated singularities:

$$\frac{1}{2\pi i}\oint_C f(z)dz = \sum_{\substack{\text{enclosed}\\ \text{residues}}} a_{-1}$$

ex. $I=\int_0^{2\pi} f(\sin\theta,\cos\theta)d\theta$

f finite single valued rational funtion of θ

$z=e^{i\theta}, dz = ie^{i\theta}d\theta$

$I=(-i)2\pi i\sum$ residues within the unit circle

Definite integrals

Ex 1.

$$I=\int_0^{2\pi}\frac{d\theta}{1+\varepsilon\cos\theta}\qquad |\varepsilon|<1.$$

$$=-i\oint_{unit\ circle}\frac{dz}{z\left[1+\varepsilon\left(z+z^{-1}\right)/2\right]}=-i\frac{2}{\varepsilon}\oint_{unit\ circle}\frac{dz}{z^2+(2/\varepsilon)z+1}=-i\frac{2}{\varepsilon}2\pi i\left.\frac{1}{2z+2/\varepsilon}\right|_{z=z_-}=\frac{2\pi}{\sqrt{1-\varepsilon^2}}$$

Ex 2.

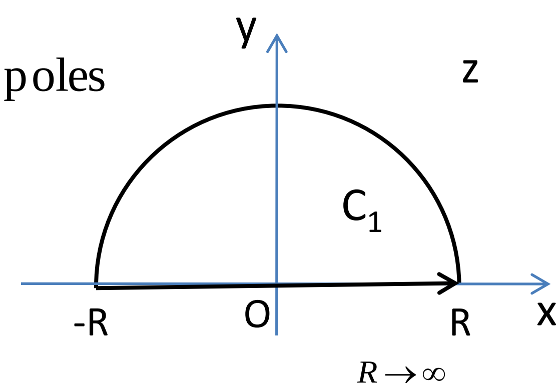
$$I=\int_{-\infty}^{\infty}f(x)dx$$

f(z) analytic in the upper half plane except for finite poles

f(z) vanishes faster than 1/z, and single valued.

$$\oint f(z)dz=\lim_{R\to\infty}\int_{-R}^{R}f(x)dx+\lim_{R\to\infty}\int_0^{\pi}f(e^{i\theta}R)ie^{i\theta}Rd\theta$$

$$I=2\pi i\sum\text{residues in upper half plane}$$



$$I=\int_{-\infty}^{\infty}\frac{dx}{1+x^2}$$

$$f(z)=\frac{1}{1+z^2}=\frac{1}{z+i}\frac{1}{z-i},\qquad a_{-1}(z_+)=(z-z_0)f(z)\Big|_{z=z_0}=1/2i\quad\Rightarrow\quad I=2\pi i\frac{1}{2i}=\pi$$

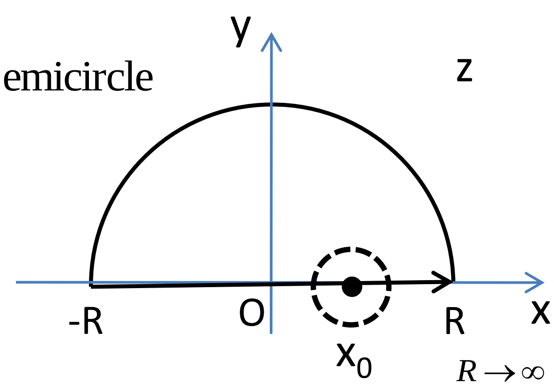
Cauchy principle value

Isolated first order pole directly on the contour of integration

$$\oint f(z)dz = \int_{-\infty}^{x_0-\delta} f(x)dx + \int_{C_{x_0}} f(z)dz + \int_{x_0+\delta}^{\infty} f(x)dx + \int_C \text{infinite semicircle}$$

$$= 2\pi i \sum \text{enclosed residues}$$

$$\int_{C_{x_0}} f(z)dz = \pm i\pi a_{-1}$$



Define

$$P\int_{-\infty}^{\infty} f(x)dx = \lim_{\delta \to 0}\left\{\int_{-\infty}^{x_0-\delta} f(x)dx + \int_{x_0+\delta}^{\infty} f(x)dx\right\}$$

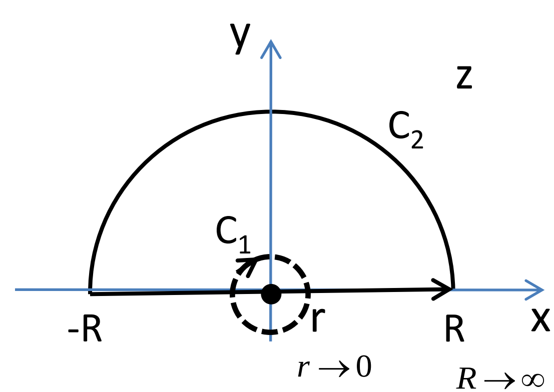
Ex .

$$I = \int_{-\infty}^{\infty} \frac{\sin x dx}{x}, \text{ the imaginary part of } I_z = P\int_{-\infty}^{\infty} \frac{e^{iz}dz}{z}$$

$$\oint \frac{e^{iz}dz}{z} = \int_{-R}^{-r} \frac{e^{ix}dx}{x} + \int_{C_1} \frac{e^{iz}dz}{z} + \int_{-r}^{R} \frac{e^{ix}dx}{x} + \int_{C_2} \frac{e^{iz}dz}{z} = 0$$

$$\int_{C_1} \frac{e^{iz}dz}{z} + P\int_{-\infty}^{\infty} \frac{e^{ix}dx}{x} = 0 \Rightarrow P\int_{-\infty}^{\infty} \frac{e^{ix}dx}{x} = i\pi$$

$$\therefore I = \int_{-\infty}^{\infty} \frac{\sin x dx}{x} = \pi$$



Dispersion relations

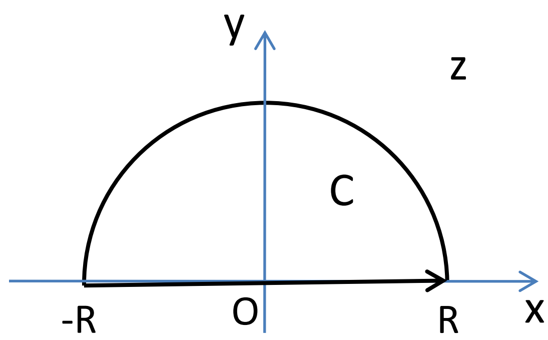
Cauchy integral formula:

$f(z)$ analytic in the upper half plane and on the real axis.

$$\lim_{|z|\to\infty} |f(z)| = 0,\ 0 \le \arg z \le \pi$$

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}dz = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(x)}{x - z_0}dx$$

$$f(x_0) = \frac{1}{\pi i}P\int_{-\infty}^{\infty} \frac{f(x)}{x - x_0}dx$$



For the complex function f(x):

$$f(x_0) = u(x_0) + iv(x_0) = \frac{1}{\pi}P\int_{-\infty}^{\infty} \frac{v(x)}{x - x_0}dx - \frac{i}{\pi}P\int_{-\infty}^{\infty} \frac{u(x)}{x - x_0}dx$$

Dispersion relations:

(Hilbert transforms)

$$u(x_0) = \frac{1}{\pi}P\int_{-\infty}^{\infty} \frac{v(x)}{x - x_0}dx$$

$$v(x_0) = -\frac{1}{\pi}P\int_{-\infty}^{\infty} \frac{u(x)}{x - x_0}dx$$

Symmetric form: $f(-x) = f^*(x)$

$$u(x_0) = \frac{2}{\pi}P\int_0^{\infty} \frac{xv(x)}{x^2 - x_0^2}dx$$

$$v(x_0) = -\frac{2}{\pi}P\int_{-\infty}^{\infty} \frac{x_0 u(x)}{x^2 - x_0^2}dx$$

Homework

- Chap 7:

7.2.7, 7.2.12, 7.2.15, 7.2.18, 7.2.19, 7.2.22, 7.2.24