Chapter 6: integral transforms

1. Green's function
2. Integral transforms
3. Integral equations
4. Examples

Green's function

Homogeneous:

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0$$

Nonhomogeneous:

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=F(x)$$

Solutions:

$$y(x)=c_1y_1(x)+c_2y_2(x)+y_p(x)$$

Green's function:

A solution to Poisson's equation with a point source

$$\nabla^2 G=-\delta(\mathbf{r}_1-\mathbf{r}_2)$$

Green's theorem

$$\int\left(\psi\nabla^2 G-G\nabla^2\psi\right)d\tau_2=\int\left(\psi\nabla G-G\nabla\psi\right)\cdot d\sigma=0$$

$$\Rightarrow\int\psi\nabla^2 G\,d\tau_2=\int G\nabla^2\psi\,d\tau_2$$

Poisson's equation

$$\nabla^2\psi=-\frac{\rho}{\varepsilon_0}$$

$$\Longrightarrow\quad \psi(\mathbf{r}_1)=\frac{1}{\varepsilon_0}\int G(\mathbf{r}_1,\mathbf{r}_2)\rho(\mathbf{r}_2)d\tau_2$$

Source and potential:

$$G(\mathbf{r}_1,\mathbf{r}_2)=\frac{1}{4\pi|\mathbf{r}_1-\mathbf{r}_2|}\qquad \psi(\mathbf{r}_1)=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{r}_2)}{|\mathbf{r}_1-\mathbf{r}_2|}d\tau_2$$

Ex. Electrostatic potential

$$\psi=\frac{1}{4\pi\varepsilon_0}\sum_i\frac{q_i}{r_i}\rightarrow\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{r})}{r}d\tau$$

General 2nd order linear nonhomogeneous differential equation

$$Ly(\mathbf{r}_1)=-f(\mathbf{r}_1)\qquad LG(\mathbf{r}_1,\mathbf{r}_2)=-\delta(\mathbf{r}_1-\mathbf{r}_2)\qquad y(\mathbf{r}_1)=\int G(\mathbf{r}_1,\mathbf{r}_2)f(\mathbf{r}_2)d\tau_2$$

Green's function

Examples:

	Laplace ∇^2	Helmholtz $\nabla^2 + k^2$	Modified Helmholtz $\nabla^2 - k^2$
3 D space $G(\mathbf{r}_1,\mathbf{r}_2)$	$\frac{1}{4\pi\lvert\mathbf{r}_1-\mathbf{r}_2\rvert}$	$\frac{\exp(ik\lvert\mathbf{r}_1-\mathbf{r}_2\rvert)}{4\pi\lvert\mathbf{r}_1-\mathbf{r}_2\rvert}$	$\frac{\exp(-k\lvert\mathbf{r}_1-\mathbf{r}_2\rvert)}{4\pi\lvert\mathbf{r}_1-\mathbf{r}_2\rvert}$

Quantum scattering:

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r})+V(\mathbf{r})\psi(\mathbf{r})=E\psi(\mathbf{r}) \qquad \nabla^2\psi(\mathbf{r})+k^2\psi(\mathbf{r})=-\left[-\frac{2m}{\hbar^2}V(\mathbf{r})\right]\psi(\mathbf{r}),\quad k^2=\frac{2mE}{\hbar^2}$$

$$\psi(\mathbf{r})\sim \underbrace{e^{i\mathbf{k}_0\cdot\mathbf{r}}}_{\text{no scattering}}+\underbrace{f_k(\theta,\varphi)\frac{e^{ikr}}{r}}_{\text{outgoing scattered}} \qquad \psi(\mathbf{r})\sim e^{i\mathbf{k}_0\cdot\mathbf{r}}-\int\frac{2m}{\hbar^2}V(\mathbf{r}_2)\psi(\mathbf{r}_2)G(\mathbf{r}_1,\mathbf{r}_2)d^3r_2$$

The solution:

$$\psi(\mathbf{r}_1)\sim e^{i\mathbf{k}_0\cdot\mathbf{r}_1}-\int\frac{2m}{\hbar^2}V(\mathbf{r}_2)\psi(\mathbf{r}_2)\frac{\exp(ik\lvert\mathbf{r}_1-\mathbf{r}_2\rvert)}{4\pi\lvert\mathbf{r}_1-\mathbf{r}_2\rvert}d^3r_2$$

Neumann series:

0-th order $\psi_0(\mathbf{r}_1)=e^{i\mathbf{k}_0\cdot\mathbf{r}_1}$

1^{st} order $\psi(\mathbf{r}_1)\sim e^{i\mathbf{k}_0\cdot\mathbf{r}_1}-\int\frac{2m}{\hbar^2}V(\mathbf{r}_2)e^{i\mathbf{k}_0\cdot\mathbf{r}_2}\frac{\exp(ik\lvert\mathbf{r}_1-\mathbf{r}_2\rvert)}{4\pi\lvert\mathbf{r}_1-\mathbf{r}_2\rvert}d^3r_2$

北京大学

Eigen-function expansion

Inhomogeneous Helmholtz equation:

$$\nabla^2\psi(\mathbf{r})+k^2\psi(\mathbf{r})=-\rho(\mathbf{r})$$

Homogeneous:

$$\nabla^2\varphi_n(\mathbf{r})+k_n^2\varphi_n(\mathbf{r})=0$$

Green's function:

$$\nabla^2 G(\mathbf{r}_1,\mathbf{r}_2)+k^2 G(\mathbf{r}_1,\mathbf{r}_2)=-\delta(\mathbf{r}_1-\mathbf{r}_2)$$

Eigen-function expansion:

$$G(\mathbf{r}_1,\mathbf{r}_2)=\sum_{n=0}^{\infty}a_n(\mathbf{r}_2)\varphi_n(\mathbf{r}_1)$$

$$G(\mathbf{r}_1,\mathbf{r}_2)=\sum_{n=0}^{\infty}\frac{\varphi_n(\mathbf{r}_1)\varphi_n(\mathbf{r}_2)}{k_n^2-k^2}$$

General inhomogeneous differential equation

$$L\psi+\lambda\psi=-\rho$$

Green's function satisfies:

$$LG(\mathbf{r}_1,\mathbf{r}_2)+\lambda G(\mathbf{r}_1,\mathbf{r}_2)=-\delta(\mathbf{r}_1-\mathbf{r}_2)$$

Eigen-function expansion:

$$G(\mathbf{r}_1,\mathbf{r}_2)=\sum_{n=0}^{\infty}\frac{\varphi_n(\mathbf{r}_1)\varphi_n(\mathbf{r}_2)}{\lambda_n^2-\lambda^2}$$

The solution to the differential equation:

$$\psi(\mathbf{r}_1)=\int G(\mathbf{r}_1,\mathbf{r}_2)\rho(\mathbf{r}_2)d\tau_2$$

Strum-Liouville ODEs

S-L equation:

$$Ly(x)+f(x)=0$$

$$L=\frac{d}{dx}\left[p(x)\frac{d}{dx}\right]+q(x)$$

Define:

$$LG_1(x)=0,\qquad \mathrm{a}\le \mathrm{x}<\mathrm{t}$$

$$LG_2(x)=0,\qquad \mathrm{t}<\mathrm{x}\le \mathrm{b}$$

Boundary conditions:

$$y(a)=0, \text{or } y'(a)=0, \text{or } \alpha y(a)+\beta y'(a)$$

$$y(b)=0, \text{or } y'(b)=0, \text{or } \alpha y(b)+\beta y'(b)$$

$$\lim_{x\to t_-} G_1(x)=\lim_{x\to t_+} G_2(x)$$

$$\left.\frac{d}{dx}G_2(x)\right|_t-\left.\frac{d}{dx}G_1(x)\right|_t=-\frac{1}{p(t)}$$

$$\Longrightarrow \quad y(x)=\int_a^b G(x,t)f(t)dt$$

Construction of Green's function:

$$G(x,t)=\begin{cases} c_1u(x), & \mathrm{a}\le \mathrm{x}<\mathrm{t} \\ c_2v(x), & \mathrm{t}<\mathrm{x}\le \mathrm{b}\end{cases}$$

$$c_2v(t)-c_1u(t)=0$$

$$c_2v'(t)-c_1u'(t)=-\frac{1}{p(t)}$$

Unique solution if:

$$\begin{vmatrix} u(t) & v(t) \\ u'(t) & v'(t)\end{vmatrix}=u(t)v'(t)-v(t)u'(t)=\frac{A}{p(t)}$$

Strum-Liouville ODEs

Green's function for S-L equation:

$$G(x,t)=\begin{cases}-\frac{1}{A}u(x)v(t), & \mathrm{a}\le \mathrm{x}<\mathrm{t}\\ -\frac{1}{A}u(t)v(x), & \mathrm{t}<\mathrm{x}\le \mathrm{b}\end{cases}$$

$$Ly(x)+f(x)=0 \qquad L=\frac{d}{dx}\left[p(x)\frac{d}{dx}\right]+q(x)$$

$$y(x)=\int_a^b G(x,t)f(t)dt$$

solution?

Proof:

$$y(x)=-\frac{1}{A}\int_a^x v(x)u(t)f(t)dt-\frac{1}{A}\int_x^b u(x)v(t)f(t)dt$$

$$y'(x)=-\frac{1}{A}\int_a^x v'(x)u(t)f(t)dt-\frac{1}{A}\int_x^b u'(x)v(t)f(t)dt$$

$$y''(x)=-\frac{1}{A}\int_a^x v''(x)u(t)f(t)dt-\frac{1}{A}\int_x^b u''(x)v(t)f(t)dt$$

$$-\frac{1}{A}\left[u(x)v'(x)-v(x)u'(x)\right]f(x)$$

$$\Longrightarrow \quad Ly(x)+f(x)=0$$

u(x), v(x) satisfies the homogeneous S-L equation

Boundary conditions:

$$y(a)=-\frac{u(a)}{A}\int_a^b v(t)f(t)dt=cu(a)$$

$$y'(a)=-\frac{u'(a)}{A}\int_a^b v(t)f(t)dt=cu'(a)$$

$$\alpha u(a)-\beta u'(a)=0$$

Equivalence

Differential equation -> integral equation

$$Ly(x)+f(x)=0$$

$$L=\frac{d}{dx}\left[p(x)\frac{d}{dx}\right]+q(x)$$

$$G(x,t)=\begin{cases}G_1(x,t), & a\le x<t\\ G_2(x,t), & t<x\le b\end{cases}$$

$$\lim_{x\to t_-}G_1(x)=\lim_{x\to t_+}G_2(x)$$

$$\left.\frac{d}{dx}G_2(x)\right|_t-\left.\frac{d}{dx}G_1(x)\right|_t=-\frac{1}{p(t)}$$

(dis)continuity

$$\xrightarrow{?}\quad y(x)=\int_a^b G(x,t)f(t)dt$$

Proof:

$$-\int_a^b G(x,t)Ly(x)dx=\int_a^b G(x,t)f(x)dx$$

$$-\int_a^t G_1(x,t)Ly(x)dx$$

$$=-\left|G_1(x,t)p(x)y'(x)\right|_a^t+\int_a^t G_1'(x,t)p(x)y'(x)dx-\int_a^t G_1(x,t)q(x)y(x)dx$$

$$=-\left|G_1(x,t)p(x)y'(x)\right|_a^t+\left|G_1'(x,t)p(x)y(x)\right|_a^t-\int_a^t y(x)LG_1(x,t)dx$$

$$LHS=-\left|G_1(x,t)p(x)y'(x)\right|_a^t+\left|G_1'(x,t)p(x)y(x)\right|_a^t$$

$$+\left|G_2(x,t)p(x)y'(x)\right|_t^b+\left|G_2'(x,t)p(x)y(x)\right|_t^b$$

$$=y(t)$$

Same boundary conditions for G(x,t) and y(x) and (dis)continuity for G and G'

$$t\longleftrightarrow x$$

$$y(x)=\int_a^b G(x,t)f(t)dt$$

Examples

Linear oscillator: $y''(x)+\lambda y(x)=0$ $\quad y(0)=y(1)=0$

To construct Green's function:

$$G(x,t)=\begin{cases}-\frac{1}{A}u(x)v(t), & 0\le x<t\\ -\frac{1}{A}u(t)v(x), & t<x\le 1\end{cases}$$

$$W=\begin{vmatrix}u(t) & v(t)\\ u'(t) & v'(t)\end{vmatrix}=u(t)v'(t)-v(t)u'(t)=\frac{A}{p(t)}$$

$$u(x)=x, v(x)=1-x$$

$$p(x)=1, W=-1\Rightarrow A=-1$$

$$\therefore\ G(x,t)=\begin{cases}x(1-t), & 0\le x<t\\ t(1-x), & t<x\le 1\end{cases}$$

$$y(x)=\lambda\int_0^1 G(x,t)y(t)dt$$

The solution of differential equation:

$$y(x)=\sin(n\pi x), \lambda=n^2\pi^2$$

$$\overset{?}{\Longrightarrow}\quad y(x)=\lambda\int_0^1 G(x,t)y(t)dt$$

Integral transforms

Green's function:

$$y(x) = \int_a^b G(x,t) f(t) dt$$

General transforms:

$$g(\alpha) = \int_a^b K(\alpha,t) f(t) dt$$

Most useful transforms:

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt, \quad \text{Fourier}$$

$$g(\alpha) = \int_0^{\infty} f(t) e^{-\alpha t} dt, \quad \text{Laplace}$$

$$g(\alpha) = \int_0^{\infty} f(t) t J_n(\alpha t) dt, \quad \text{Hankel}$$

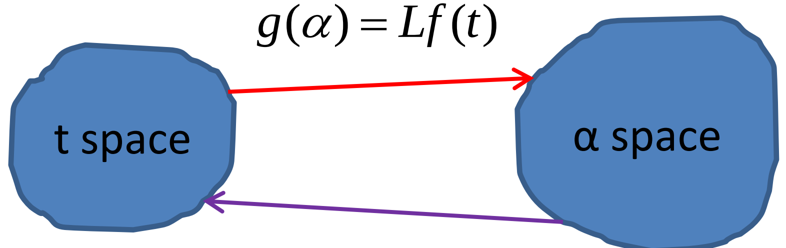
$$g(\alpha) = \int_0^{\infty} f(t) t^{\alpha-1} dt, \quad \text{Mellin}$$

$$g(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \cos(\omega t) dt, \quad \text{Cosine}$$

$$g(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \sin(\omega t) dt, \quad \text{Sine}$$

Linear transforms:

$$g(\alpha) = L f(t)$$



Inverse transforms:

$$f(t) = L^{-1} g(\alpha)$$

Ex:

$$g(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} f(\mathbf{r}) e^{i\mathbf{k}\cdot\mathbf{r}} d^3 r$$

$$f(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} g(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{r}} d^3 k$$

Fourier transforms

Fourier integral theorem:

$$f(x)=\frac{1}{\pi}\int_0^{\infty}d\omega\int_{-\infty}^{\infty}f(t)\cos[\omega(t-x)]dt$$

f(x) is (1) piecewise continuous, (2) differentiable, (3) absolutely integrable.

Exponential form:

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\omega x}d\omega\int_{-\infty}^{\infty}f(t)e^{i\omega t}dt$$

Fourier transforms:

$$g(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)e^{i\omega t}dt$$

Inverse transforms:

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}g(\omega)e^{-i\omega x}d\omega$$

Delta function: $\delta(t-x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\omega(t-x)}d\omega$

Fourier transforms of derivatives:

$$g_1(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{df(t)}{dt}e^{i\omega t}dt=-i\omega g(\omega)$$

$$g_n(\omega)=(-i\omega)^n g(\omega)$$

北京大学

Applications

1-D wave equation:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} + H(x,t) \\ u(x,0) = f(x),\ \dfrac{\partial u(x,0)}{\partial t} = g(x) \end{cases}$$

Fourier transforms:

$$\hat{u}(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t) e^{ikx} dx$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(k,t) e^{-ikx} dk$$

In k-space:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{\partial^2 \hat{u}(k,t)}{\partial t^2} + c^2 k^2 \hat{u}(k,t) - \hat{H}(k,t) \right] e^{-ikx} dk = 0 \longrightarrow \frac{\partial^2 \hat{u}(k,t)}{\partial t^2} + c^2 k^2 \hat{u}(k,t) = \hat{H}(k,t)$$

$$\therefore\ \hat{u}(k,t) = A(k)\cos(ckt) + B(k)\sin(ckt) + \frac{1}{ck}\int_0^t \hat{H}(k,t_1)\sin[ck(t-t_1)]dt_1$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left\{ f(x_1)\cos(ckt) + g(x_1)\frac{\sin(ckt)}{ck} + \int_0^t H(x_1,t_1)\frac{\sin[ck(t-t_1)]}{ck} dt_1 \right\} e^{ikx_1} dx_1$$

The solution:

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}(k,t) e^{-ikx} dk$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} dk \int_{-\infty}^{\infty} e^{ikx_1} dx_1 \left\{ f(x_1)\cos(ckt) + g(x_1)\sin(ckt) + \frac{1}{ck}\int_0^t H(x_1,t_1)\sin[ck(t-t_1)]dt_1 \right\}$$

Wave equation

The solution:

$$u(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-ikx}dk\int_{-\infty}^{\infty}e^{ikx_1}dx_1\left\{f(x_1)\cos(ckt)+g(x_1)\sin(ckt)+\frac{1}{ck}\int_0^t H(x_1,t_1)\sin[ck(t-t_1)]dt_1\right\}$$

First term:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-ikx}dk\int_{-\infty}^{\infty}e^{ikx_1}dx_1 f(x_1)\cos(ckt)$$

$$=\int_{-\infty}^{\infty}dx_1 f(x_1)\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-ikx}e^{ikx_1}\frac{1}{2}\left(e^{ickt}+e^{-ickt}\right)dk$$

$$=\frac{1}{2}f(x+ct)+\frac{1}{2}f(x-ct)$$

Second term:

$$\int_{-\infty}^{\infty}dx_1 g(x_1)\frac{1}{2\pi i}\int_{-\infty}^{\infty}e^{-ik(x-x_1)}\frac{1}{2ck}\left(e^{ickt}-e^{-ickt}\right)dk$$

$$=\frac{1}{2c}\int_{-\infty}^{\infty}dx_1 g(x_1)[h(x-x_1+ct)-h(x-x_1-ct)]$$

$$=\frac{1}{2c}\int_{x-ct}^{x+ct}dx_1 g(x_1)$$

Third term (source):

$$=\frac{1}{2c}\int_0^t dt_1\int_{x-c(t-t_1)}^{x+c(t-t_1)}dx_1 H(x_1,t_1)$$

The image here is removed due to copy right issues

Special case

(1) Point source:

$$H(x,t)=\delta(x-x_0)f(t)$$

$$u_H(x,t)=\frac{1}{2c}\int_0^t dt_1\int_{x-c(t-t_1)}^{x+c(t-t_1)}dx_1\delta(x_1-x_0)f(t_1)$$

$$f(t)=\cos(\omega t)$$

$$u_H(x,t)=\frac{1}{2c\omega}\sin\left[\omega\left(t-\frac{x-x_0}{c}\right)\right]\quad \text{for } t>\frac{x-x_0}{c}$$

Doppler's effect

(2) Moving source:

$$H(x,t)=\delta(x-x_0-vt)\cos(\omega t)$$

$$u_H(x,t)=\frac{1}{2c}\int_0^t dt_1\int_{x-c(t-t_1)}^{x+c(t-t_1)}dx_1\delta(x_1-x_0-vt_1)\cos(\omega t_1)$$

For v < c

$$u_H(x,t)=\begin{cases}0, & t<\frac{x\text{-}x_0}{c}\\ \frac{1}{2\omega c}\sin\left(\frac{\omega c}{c-v}t-\omega\frac{x-x_0}{c-v}\right), & \frac{x\text{-}x_0}{c}<t<\frac{x\text{-}x_0}{v}\\ \frac{1}{2\omega c}\sin\left(\frac{\omega c}{c+v}t+\omega\frac{x-x_0}{c+v}\right), & \frac{x\text{-}x_0}{v}<t\end{cases}$$

Images here are removed due to copy right issues

Red shift

For v > c

$$u_H(x,t)=\begin{cases}0, & t<\frac{x\text{-}x_0}{v}\\ \frac{1}{2\omega c}\left[\sin\left(\frac{\omega c}{c+v}t+\omega\frac{x-x_0}{c+v}\right)+\sin\left(\frac{\omega c}{c-v}t-\omega\frac{x-x_0}{c-v}\right)\right], & \frac{x\text{-}x_0}{v}<t<\frac{x\text{-}x_0}{c}\\ \frac{1}{2\omega c}\sin\left(\frac{\omega c}{c+v}t+\omega\frac{x-x_0}{c+v}\right), & \frac{x\text{-}x_0}{c}<t\end{cases}$$

Laplace transforms

Definition:

$$f(s)=L[F(t)]=\int_0^{\infty}F(t)e^{-st}dt\,,\quad \text{Laplace}$$

$$F(t)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}f(s)e^{st}ds\,,\quad \text{inverse transform}$$

Linear transforms:

$$F(t)=L^{-1}[f(s)]$$

Examples:

$$L[1]=\int_0^{\infty}e^{-st}dt=\frac{1}{s},\quad s>0$$

$$L[e^{kt}]=\int_0^{\infty}e^{kt}e^{-st}dt=\frac{1}{s-k},\quad s>k$$

$$L[\cos kt]=\int_0^{\infty}\cos kte^{-st}dt=\frac{s}{s^2+k^2}$$

$$L[\sin kt]=\int_0^{\infty}\sin kte^{-st}dt=\frac{k}{s^2+k^2}$$

Derivatives:

$$L[F'(t)]=\int_0^{\infty}\frac{dF(t)}{dt}e^{-st}dt=e^{-st}F(t)\Big|_0^{\infty}+s\int_0^{\infty}F(t)e^{-st}dt=sL[F(t)]-F(0)$$

Harmonic oscillator:

$$m\frac{d^2X(t)}{dt^2}+kX(t)=0\,,\quad X(0)=X_0\,,\,X'(0)=0$$

$$mL\left[\frac{d^2X(t)}{dt^2}\right]+kL[X(t)]=0\Rightarrow ms^2x(s)-msX(0)+kx(s)=0$$

$$x(s)=X_0\frac{s}{s^2+\omega_0^2}\,,\,\omega_0=\frac{k}{m}$$

$$X(t)=X_0\cos(\omega_0 t)$$